

INITIAL-BOUNDARY VALUE PROBLEM FOR INTEGRABLE NONLINEAR EVOLUTION EQUATIONS WITH 3×3 LAX PAIRS ON THE INTERVAL

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ABSTRACT. We present an approach for analyzing initial-boundary value problems which is formulated on the finite interval $(0 \leq x \leq L$, where L is a positive constant) for integrable equations whose Lax pairs involve 3×3 matrices. Boundary value problems for integrable nonlinear evolution PDEs can be analyzed by the unified method introduced by Fokas and developed by him and his collaborators. In this paper, we show that the solution can be expressed in terms of the solution of a 3×3 Riemann-Hilbert problem. The relevant jump matrices are explicitly given in terms of the three matrix-value spectral functions $s(k), S(k)$ and $S_L(k)$, which in turn are defined in terms of the initial values, boundary values at $x = 0$ and boundary values at $x = L$, respectively. However, these spectral functions are not independent, they satisfy a global relation. Here, we show that the characterization of the unknown boundary values in terms of the given initial and boundary data is explicitly described for a nonlinear evolution PDE defined on the interval. Also, we show that in the limit when the length of the interval tends to infinity, the relevant formulas reduce to the analogous formulas obtained for the case of boundary value problems formulated on the half-line.

1. INTRODUCTION

Integrable PDEs have the distinctive property that they can be written as the compatibility condition of two linear eigenvalue equations,

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which are called a Lax pair [1]. An effective method (Inverse Scattering Transform (IST)) for solving the initial value problem for integrable evolution equations on the line was discovered in 1967 [2]. However, the presence of a boundary presents new challenges. It was realized in [3] that the extension of this method to initial boundary value problems requires a deeper understanding of the following question: What is the fundamental transform for solving initial boundary value problems for linear evolution equations with x -derivatives of arbitrary order? The investigation of this question has led to the discovery of a general approach for solving boundary value problems for linear and for integrable nonlinear PDEs [4] (see also [5, 6]). For integrable nonlinear evolution PDEs this approach is based on the simultaneous spectral analysis of the two linear eigenvalue equations forming the Lax pair, and on the investigation of the so-called global relation, which is an algebraic relation coupling the relevant spectral functions.

The Fokas method provides a generalization of the IST formalism from initial value to initial-boundary value (IBV) problems, and over the last eighteen years, this method has been used to analyze boundary value problems for several of the most important integrable equations with 2×2 Lax pairs, such as the Korteweg-de Vries [7], the nonlinear Schrödinger [8], the sine-Gordon equations [9], see [10, 11, 12, 13]. Just like the IST on the line, the unified method yields an expression for the solution of an IBV problem in terms of the solution of a Riemann-Hilbert problem. In particular, the asymptotic behavior of the solution can be analyzed in an effective way by using this Riemann-Hilbert problem and by employing the nonlinear version of the steepest descent method introduced by Deift and Zhou [17].

In 2012, Lenells first develops a methodology for analyzing IBV problems on the half-line for integrable evolution equations with Lax pairs involving 3×3 matrices [18]. Although the transition from 2×2 to 3×3 matrix Lax pairs involves a number of novelties, the two main steps of the method of [3, 6] remain the same:

(1) Construct an integral representation of the solution characterized via a matrix Riemann-Hilbert problem formulated on the complex k -sphere, where k denotes the spectral parameter of the Lax pair. This representation involves, in general, some unknown boundary values, thus the solution formula is not yet effective.

(2) Characterize the unknown boundary values by analyzing the so-called global relation. In general, the characterization of the unknown boundary values involves the solution of a nonlinear problem.

After Lenells' work, IBV problems on the half-line for other integrable evolution equations such as the Degasperis-Procesi [19], Sasa-Satsuma [20], three wave [21], the two-component nonlinear Schrödinger [22], the Ostrovsky-Vakhnenko [23] equations, are analyzed. However, within the knowledge of the authors, the IBV problems for integrable equations with 3×3 matrices Lax pair on the finite interval has not been studied yet.

The purpose of this paper is to extend the ideas from analyzing the IBV problems on the half-line to the finite interval for integrable evolution equations with Lax pairs involving 3×3 matrices. In fact, dealing with IBV problems on the interval has some difficulties. The implementation of step (1), we need four curve integration from the four corners of the (x, t) -domain. We will define analytic eigenfunctions, denoted by $\{M_n(x, t, k)\}$, via integral equations which involve integration from all *four* corners simultaneously. The most difficulties is to make a distinction between the integration contour γ_3 and γ_4 when we try to analyze the IBV problems on the interval. It is different from the analyzing the IBV problems on the half-line, because in the half-line case there just one integration curve γ_3 . Here, the constructions of this paper can be compared with the corresponding formalism for 2×2 -matrix Lax pairs introduced by Fokas and Its, see [11]. The implementation of step (2), the differences are introducing a new factor $\frac{1}{\Delta}$ during analyzing the global relation to characterize the unknown boundary data in terms of the given initial and boundary data. We

show that in the limit when the length of the interval tends to infinity, the relevant formulas reduce to the analogous formulas obtained for the case of boundary value problems formulated on the half-line.

Organization of the paper: In the next subsection, we introduce our main example considered in this paper. In section 2 we perform the spectral analysis of the associated Lax pair. We formulate the main Riemann-Hilbert problem in section 3 and this concludes the implementation of step (1) above. We also get the map between the Dirichlet and the Neumann boundary problem through analyzing the global relation in section 4 and this concludes the implementation of step (2).

1.1. The main example. In this paper, we will consider the two-component nonlinear Schrödinger equation or Manakov equation

$$\begin{cases} iq_{1t} + q_{1xx} - 2\sigma(|q_1|^2 + |q_2|^2)q_1 = 0, \\ iq_{2t} + q_{2xx} - 2\sigma(|q_1|^2 + |q_2|^2)q_2 = 0. \end{cases} \quad \sigma = \pm 1. \quad (1.1)$$

where $q_1(x, t)$ and $q_2(x, t)$ are complex-valued functions of $(x, t) \in \Omega$, with Ω denoting the finite interval domain

$$\Omega = \{(x, t) | 0 \leq x \leq L, 0 \leq t \leq T\}, \quad (1.2)$$

here $L > 0$ is a positive fixed constant and $T > 0$ being a fixed final time. Here, $\sigma = 1$ means defocusing case and $\sigma = -1$ means focusing case. This system was first introduced by Manakov to describe the propagation of an optical pulse in a birefringent optical fiber [24]. Subsequently, this system also arises in the context of multicomponent Bose-Einstein condensates [25].

We will consider the following initial-boundary value problem for the 2-NLS equation,

$$\begin{aligned}
\text{Initial value:} \quad & q_{10}(x) = q_1(x, t = 0), \quad q_{20}(x) = q_2(x, t = 0), \\
\text{Dirichlet boundary value:} \quad & g_{01}(t) = q_1(x = 0, t), \quad g_{02}(t) = q_2(x = 0, t), \\
& f_{01}(t) = q_1(x = L, t), \quad f_{02}(t) = q_2(x = L, t), \\
\text{Neumann boundary value:} \quad & g_{11}(t) = q_{1x}(x = 0, t), \quad g_{12}(t) = q_{2x}(x = 0, t), \\
& f_{11}(t) = q_{1x}(x = L, t), \quad f_{12}(t) = q_{2x}(x = L, t).
\end{aligned} \tag{1.3}$$

It is well known that 2-NLS equation admits a 3×3 Lax pair,

$$\Psi_x = U\Psi, \quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}. \tag{1.4a}$$

$$\Psi_t = V\Psi. \tag{1.4b}$$

where

$$U = ik\Lambda + V_1. \tag{1.5}$$

and

$$V = 2ik^2\Lambda + V_2 \tag{1.6}$$

here

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, V_1 = \begin{pmatrix} 0 & q_1 & q_2 \\ \sigma\bar{q}_1 & 0 & 0 \\ \sigma\bar{q}_2 & 0 & 0 \end{pmatrix}, V_2 = 2kV_2^{(1)} + V_2^{(0)}. \tag{1.7}$$

where

$$V_2^{(1)} = V_1, \quad V_2^{(0)} = i\Lambda(V_1^2 - V_{1x}). \tag{1.8}$$

2. SPECTRAL ANALYSIS

2.1. The closed one-form. Introducing a new eigenfunction $\mu(x, t, k)$ by

$$\Psi = \mu e^{i\Lambda kx + 2i\Lambda k^2 t} \tag{2.1}$$

then we find the Lax pair equations

$$\begin{cases} \mu_x - [ik\Lambda, \mu] = V_1\mu, \\ \mu_t - [2ik^2\Lambda, \mu] = V_2\mu. \end{cases} \quad (2.2)$$

Letting \hat{A} denotes the operators which acts on a 3×3 matrix X by $\hat{A}X = [A, X]$, then the equations in (2.2) can be written in differential form as

$$d(e^{-(ikx+2ik^2t)\hat{\Lambda}}\mu) = W, \quad (2.3)$$

where $W(x, t, k)$ is the closed one-form defined by

$$W = e^{-(ikx+2ik^2t)\hat{\Lambda}}(V_1dx + V_2dt)\mu. \quad (2.4)$$

2.2. The μ_j 's definition. We define four eigenfunctions $\{\mu_j\}_1^4$ of (2.2) by the Volterra integral equations

$$\mu_j(x, t, k) = \mathbb{I} + \int_{\gamma_j} e^{(ikx+2ik^2t)\hat{\Lambda}} W_j(x', t', k). \quad j = 1, 2, 3, 4. \quad (2.5)$$

where W_j is given by (2.4) with μ replaced with μ_j , and the contours $\{\gamma_j\}_1^4$ are showed in Figure 1. The first, second and third column of

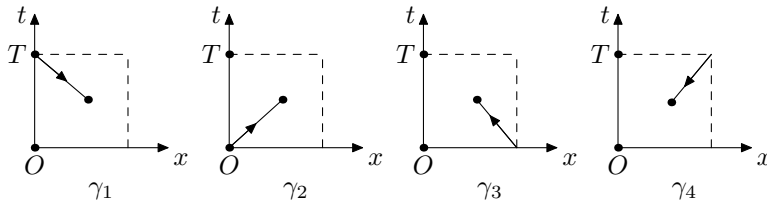


FIGURE 1. The three contours $\gamma_1, \gamma_2, \gamma_3$ and γ_4 in the (x, t) -domain.

the matrix equation (2.5) involves the exponentials

$$\begin{aligned} [\mu_j]_1: & e^{2ik(x-x')+4ik^2(t-t')}, e^{2ik(x-x')+4ik^2(t-t')} \\ [\mu_j]_2: & e^{-2ik(x-x')-4ik^2(t-t')}, \\ [\mu_j]_3: & e^{-2ik(x-x')-4ik^2(t-t')}. \end{aligned} \quad (2.6)$$

And we have the following inequalities on the contours:

$$\begin{aligned}
 \gamma_1 : \quad & x - x' \geq 0, t - t' \leq 0, \\
 \gamma_2 : \quad & x - x' \geq 0, t - t' \geq 0, \\
 \gamma_3 : \quad & x - x' \leq 0, t - t' \geq 0, \\
 \gamma_4 : \quad & x - x' \leq 0, t - t' \leq 0.
 \end{aligned} \tag{2.7}$$

So, these inequalities imply that the functions $\{\mu_j\}_1^4$ are bounded and analytic for $k \in \mathbb{C}$ such that k belongs to

$$\begin{aligned}
 \mu_1 : \quad & (D_2, D_3, D_3), \\
 \mu_2 : \quad & (D_1, D_4, D_4), \\
 \mu_3 : \quad & (D_3, D_2, D_2), \\
 \mu_4 : \quad & (D_4, D_1, D_1).
 \end{aligned} \tag{2.8}$$

where $\{D_n\}_1^4$ denote four open, pairwise disjoint subsets of the complex k -sphere showed in Figure 2.

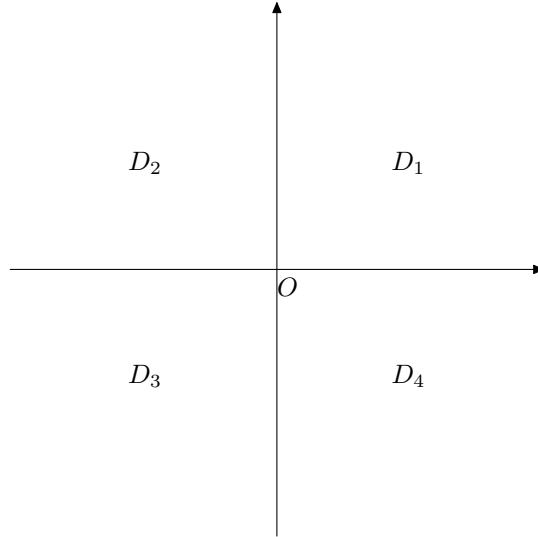


FIGURE 2. The sets D_n , $n = 1, \dots, 4$, which decompose the complex k -plane.

We also notice that the sets $\{D_n\}_1^4$ has the following properties:

$$\begin{aligned} D_1 &= \{k \in \mathbb{C} | \text{Rel}_1 > \text{Rel}_2 = \text{Rel}_3, \text{Rez}_1 > \text{Rez}_2 = \text{Rez}_3\}, \\ D_2 &= \{k \in \mathbb{C} | \text{Rel}_1 > \text{Rel}_2 = \text{Rel}_3, \text{Rez}_1 < \text{Rez}_2 = \text{Rez}_3\}, \\ D_3 &= \{k \in \mathbb{C} | \text{Rel}_1 < \text{Rel}_2 = \text{Rel}_3, \text{Rez}_1 > \text{Rez}_2 = \text{Rez}_3\}, \\ D_4 &= \{k \in \mathbb{C} | \text{Rel}_1 < \text{Rel}_2 < \text{Rel}_3, \text{Rez}_1 < \text{Rez}_2 = \text{Rez}_3\}, \end{aligned}$$

where $l_i(k)$ and $z_i(k)$ are the diagonal entries of matrices $ik\Lambda$ and $2ik^2\Lambda$, respectively.

2.3. The M_n 's definition. For each $n = 1, \dots, 4$, define a solution $M_n(x, t, k)$ of (2.2) by the following system of integral equations:

$$(M_n)_{ij}(x, t, k) = \delta_{ij} + \int_{\gamma_{ij}^n} (e^{(ikx+2ik^2t)\hat{\Lambda}} W_n(x', t', k))_{ij}, \quad k \in D_n, \quad i, j = 1, 2, 3. \quad (2.9)$$

where W_n is given by (2.4) with μ replaced with M_n , and the contours γ_{ij}^n , $n = 1, \dots, 4$, $i, j = 1, 2, 3$ are defined by

$$\gamma_{ij}^n = \begin{cases} \gamma_1 & \text{if } \text{Rel}_i(k) < \text{Rel}_j(k) \text{ and } \text{Rez}_i(k) \geq \text{Rez}_j(k), \\ \gamma_2 & \text{if } \text{Rel}_i(k) < \text{Rel}_j(k) \text{ and } \text{Rez}_i(k) < \text{Rez}_j(k), \\ \gamma_3 & \text{if } \text{Rel}_i(k) \geq \text{Rel}_j(k) \text{ and } \text{Rez}_i(k) \leq \text{Rez}_j(k), \\ \gamma_4 & \text{if } \text{Rel}_i(k) \geq \text{Rel}_j(k) \text{ and } \text{Rez}_i(k) \geq \text{Rez}_j(k). \end{cases} \quad \text{for } k \in D_n. \quad (2.10)$$

Here, we make a distinction between the contours γ_3 and γ_4 as follows,

$$\gamma_{ij}^n = \begin{cases} \gamma_3, & \text{if } \prod_{1 \leq i < j \leq 3} (\text{Rel}_i(k) - \text{Rel}_j(k))(\text{Rez}_i(k) - \text{Rez}_j(k)) < 0, \\ \gamma_4, & \text{if } \prod_{1 \leq i < j \leq 3} (\text{Rel}_i(k) - \text{Rel}_j(k))(\text{Rez}_i(k) - \text{Rez}_j(k)) > 0. \end{cases} \quad (2.11)$$

The rule chosen in the produce is if $l_m = l_n$, m may not equals n , we just choose the subscript is smaller one.

According to the definition of the γ^n , we find that

$$\begin{aligned} \gamma^1 &= \begin{pmatrix} \gamma_4 & \gamma_4 & \gamma_4 \\ \gamma_2 & \gamma_4 & \gamma_4 \\ \gamma_2 & \gamma_4 & \gamma_4 \end{pmatrix} & \gamma^2 &= \begin{pmatrix} \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_3 & \gamma_3 \end{pmatrix} \\ \gamma^3 &= \begin{pmatrix} \gamma_3 & \gamma_1 & \gamma_1 \\ \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_3 & \gamma_3 & \gamma_3 \end{pmatrix} & \gamma^4 &= \begin{pmatrix} \gamma_4 & \gamma_2 & \gamma_2 \\ \gamma_4 & \gamma_4 & \gamma_4 \\ \gamma_4 & \gamma_4 & \gamma_4 \end{pmatrix}. \end{aligned} \quad (2.12)$$

The following proposition ascertains that the M_n 's defined in this way have the properties required for the formulation of a Riemann-Hilbert problem.

Proposition 2.1. *For each $n = 1, \dots, 4$, the function $M_n(x, t, k)$ is well-defined by equation (2.9) for $k \in \bar{D}_n$ and $(x, t) \in \Omega$. For any fixed point (x, t) , M_n is bounded and analytic as a function of $k \in D_n$ away from a possible discrete set of singularities $\{k_j\}$ at which the Fredholm determinant vanishes. Moreover, M_n admits a bounded and continuous extension to \bar{D}_n and*

$$M_n(x, t, k) = \mathbb{I} + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty, \quad k \in D_n. \quad (2.13)$$

Proof. The boundedness and analyticity properties are established in appendix B in [18]. And substituting the expansion

$$M = M_0 + \frac{M^{(1)}}{k} + \frac{M^{(2)}}{k^2} + \dots, \quad k \rightarrow \infty.$$

into the Lax pair (2.2) and comparing the terms of the same order of k yield the equation (2.13). \square

2.4. The jump matrices. We define matrix-value functions $S_n(k)$, $n = 1, \dots, 4$, and

$$S_n(k) = M_n(0, 0, k), \quad k \in D_n, \quad n = 1, \dots, 4. \quad (2.14)$$

Let M denote the sectionally analytic function on the complex k -sphere which equals M_n for $k \in D_n$. Then M satisfies the jump conditions

$$M_n = M_m J_{m,n}, \quad k \in \bar{D}_n \cap \bar{D}_m, \quad n, m = 1, \dots, 4, \quad n \neq m, \quad (2.15)$$

where the jump matrices $J_{m,n}(x, t, k)$ are defined by

$$J_{m,n} = e^{(ikx+2ik^2t)\hat{\Lambda}}(S_m^{-1}S_n). \quad (2.16)$$

2.5. The adjugated eigenfunctions. We will also need the analyticity and boundedness properties of the minors of the matrices $\{\mu_j(x, t, k)\}_1^4$. We recall that the cofactor matrix X^A of a 3×3 matrix X is defined by

$$X^A = \begin{pmatrix} m_{11}(X) & -m_{12}(X) & m_{13}(X) \\ -m_{21}(X) & m_{22}(X) & -m_{23}(X) \\ m_{31}(X) & -m_{32}(X) & m_{33}(X) \end{pmatrix},$$

where $m_{ij}(X)$ denote the (ij) th minor of X .

It follows from (2.2) that the adjugated eigenfunction μ^A satisfies the Lax pair

$$\begin{cases} \mu_x^A + [ik\Lambda, \mu^A] = -V_1^T \mu^A, \\ \mu_t^A + [2ik^2\Lambda, \mu^A] = -V_2^T \mu^A. \end{cases} \quad (2.17)$$

where V^T denote the transform of a matrix V . Thus, the eigenfunctions $\{\mu_j^A\}_1^4$ are solutions of the integral equations

$$\mu_j^A(x, t, k) = \mathbb{I} - \int_{\gamma_j} e^{-ik(x-x')-2ik^2(t-t')\hat{\Lambda}}(V_1^T dx + V_2^T) \mu^A, \quad j = 1, 2, 3, 4. \quad (2.18)$$

Then we can get the following analyticity and boundedness properties:

$$\begin{aligned} \mu_1^A &: (D_3, D_2, D_2), \\ \mu_2^A &: (D_4, D_1, D_1), \\ \mu_3^A &: (D_2, D_3, D_3), \\ \mu_4^A &: (D_1, D_4, D_4). \end{aligned} \quad (2.19)$$

2.6. Symmetries. We will show that the eigenfunctions $\mu_j(x, t, k)$ satisfy an important symmetry.

Lemma 2.2. *The eigenfunction $\Psi(x, t, k)$ of the Lax pair (1.4) satisfies the following symmetry:*

$$\Psi^{-1}(x, t, k) = A \overline{\Psi(x, t, \bar{k})}^T A, \quad (2.20)$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & -\sigma \end{pmatrix}, \quad \sigma^2 = 1. \quad (2.21)$$

Here, the superscript T denotes a matrix transpose.

Proof. The equation (2.20) follows from the fact

$$-A \overline{U(x, t, \bar{k})} A = U(x, t, k)^T, \quad -A \overline{V(x, t, \bar{k})} A = V(x, t, k)^T, \quad (2.22)$$

and

$$\Psi_x^A(x, t, k) = -U(x, t, k)^T \Psi^A(x, t, k), \quad \Psi_t^A(x, t, k) = -V(x, t, k)^T \Psi^A(x, t, k) \quad (2.23)$$

□

Remark 2.3. *This lemma implies that the eigenfunctions $\mu_j(x, t, k)$ of Lax pair (2.2) satisfy the same symmetry.*

2.7. The $J_{m,n}$'s computation. Let us define the 3×3 -matrix value spectral functions $s(k)$, $S(k)$ and $S_L(k)$ by

$$\mu_3(x, t, k) = \mu_2(x, t, k) e^{(ikx + 2ik^2t)\hat{\Lambda}} s(k), \quad (2.24a)$$

$$\mu_1(x, t, k) = \mu_2(x, t, k) e^{(ikx + 2ik^2t)\hat{\Lambda}} S(k), \quad (2.24b)$$

$$\mu_4(x, t, k) = \mu_3(x, t, k) e^{(ik(x-L) + 2ik^2t)\hat{\Lambda}} S_L(k) \quad (2.24c)$$

Thus,

$$s(k) = \mu_3(0, 0, k), \quad (2.25a)$$

$$S(k) = \mu_1(0, 0, k) = e^{-2ik^2T\hat{\Lambda}} \mu_2^{-1}(0, T, k), \quad (2.25b)$$

$$S_L(k) = \mu_4(L, 0, k) = e^{-2ik^2T\hat{\Lambda}} \mu_3^{-1}(L, T, k). \quad (2.25c)$$

And we deduce from the properties of μ_j and μ_j^A that $\{s(k), S(k), S_L(k)\}$ and $\{s^A(k), S^A(k), S_L^A(k)\}$ have the following boundedness properties:

$$\begin{aligned} s(k) : & \quad (D_3 \cup D_4, D_1 \cup D_2, D_1 \cup D_2), \\ S(k) : & \quad (D_2 \cup D_4, D_1 \cup D_3, D_1 \cup D_3), \\ S_L(k) : & \quad (D_2 \cup D_4, D_1 \cup D_3, D_1 \cup D_3) \\ s^A(k) : & \quad (D_1 \cup D_2, D_3 \cup D_4, D_3 \cup D_4), \\ S^A(k) : & \quad (D_1 \cup D_3, D_2 \cup D_4, D_2 \cup D_4), \\ S_L^A(k) : & \quad (D_1 \cup D_3, D_2 \cup D_4, D_2 \cup D_4). \end{aligned}$$

Moreover, noticing that

$$M_n(x, t, k) = \mu_2(x, t, k) e^{(ikx + 2ik^2t)\hat{\Lambda}} S_n(k), \quad k \in D_n. \quad (2.26)$$

Proposition 2.4. *The S_n can be expressed in terms of the entries of $s(k), S(k)$ and $S_L(k)$ as follows:*

$$S_1 = \begin{pmatrix} \frac{1}{m_{11}(\mathcal{A})} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ 0 & \mathcal{A}_{22} & \mathcal{A}_{23} \\ 0 & \mathcal{A}_{32} & \mathcal{A}_{33} \end{pmatrix}, \quad S_2 = \begin{pmatrix} \frac{S_{11}}{(S^T s^A)_{11}} & s_{12} & s_{13} \\ \frac{S_{21}}{(S^T s^A)_{11}} & s_{22} & s_{23} \\ \frac{S_{31}}{(S^T s^A)_{11}} & s_{32} & s_{33} \end{pmatrix}, \quad (2.27a)$$

$$\begin{aligned} S_3 &= \begin{pmatrix} s_{11} & \frac{m_{33}(s)m_{21}(S) - m_{23}(s)m_{31}(S)}{(s^T S^A)_{11}} & \frac{m_{32}(s)m_{21}(S) - m_{22}(s)m_{31}(S)}{(s^T S^A)_{11}} \\ s_{21} & \frac{m_{33}(s)m_{11}(S) - m_{13}(s)m_{31}(S)}{(s^T S^A)_{11}} & \frac{m_{32}(s)m_{11}(S) - m_{12}(s)m_{31}(S)}{(s^T S^A)_{11}} \\ s_{31} & \frac{m_{23}(s)m_{11}(S) - m_{13}(s)m_{21}(S)}{(s^T S^A)_{11}} & \frac{m_{22}(s)m_{11}(S) - m_{12}(s)m_{21}(S)}{(s^T S^A)_{11}} \end{pmatrix}, \\ S_4 &= \begin{pmatrix} \mathcal{A}_{11} & 0 & 0 \\ \mathcal{A}_{21} & \frac{m_{33}(\mathcal{A})}{\mathcal{A}_{11}} & \frac{m_{32}(\mathcal{A})}{\mathcal{A}_{11}} \\ \mathcal{A}_{31} & \frac{m_{23}(\mathcal{A})}{\mathcal{A}_{11}} & \frac{m_{22}(\mathcal{A})}{\mathcal{A}_{11}} \end{pmatrix}. \end{aligned} \quad (2.27b)$$

where $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1}^3$ is a 3×3 matrix, which is defined as $\mathcal{A} = s(k) e^{-ikL\hat{\Lambda}} S_L(k)$. And the functions

$$\begin{aligned} (S^T s^A)_{11} &= S_{11}m_{11}(s) - S_{21}m_{21}(s) + S_{31}m_{31}(s), \\ (s^T S^A)_{11} &= s_{11}m_{11}(S) - s_{21}m_{21}(S) + s_{31}m_{31}(S). \end{aligned}$$

Proof. Firstly, we define $R_n(k), T_n(k)$ and $Q_n(k)$ as follows:

$$R_n(k) = e^{-2ik^2T\hat{\Lambda}} M_n(0, T, k), \quad (2.28a)$$

$$T_n(k) = e^{-ikL\hat{\Lambda}} M_n(L, 0, k), \quad (2.28b)$$

$$Q_n(k) = e^{-(ikL+2ik^2T)\hat{\Lambda}} M_n(L, T, k). \quad (2.28c)$$

Then, we have the following relations:

$$\begin{cases} M_n(x, t, k) = \mu_1(x, t, k) e^{(ikx+2ik^2t)\hat{\Lambda}} R_n(k), \\ M_n(x, t, k) = \mu_2(x, t, k) e^{(ikx+2ik^2t)\hat{\Lambda}} S_n(k), \\ M_n(x, t, k) = \mu_3(x, t, k) e^{(ikx+2ik^2t)\hat{\Lambda}} T_n(k), \\ M_n(x, t, k) = \mu_3(x, t, k) e^{(ikx+2ik^2t)\hat{\Lambda}} Q_n(k) \end{cases} \quad (2.29)$$

The relations (2.29) imply that

$$\begin{aligned} s(k) &= S_n(k) T_n^{-1}(k), \\ S(k) &= S_n(k) R_n^{-1}(k), \\ \mathcal{A}(k) &= S_n(k) Q_n^{-1}(k). \end{aligned} \quad (2.30)$$

These equations constitute a matrix factorization problem which, given $\{s(k), S(k), S_L(k)\}$ can be solved for the $\{R_n, S_n, T_n, Q_n\}$. Indeed, the integral equations (2.9) together with the definitions of $\{R_n, S_n, T_n, Q_n\}$ imply that

$$\begin{cases} (R_n(k))_{ij} = 0 & \text{if } \gamma_{ij}^n = \gamma_1, \\ (S_n(k))_{ij} = 0 & \text{if } \gamma_{ij}^n = \gamma_2, \\ (T_n(k))_{ij} = \delta_{ij} & \text{if } \gamma_{ij}^n = \gamma_3, \\ (Q_n(k))_{ij} = \delta_{ij} & \text{if } \gamma_{ij}^n = \gamma_4. \end{cases} \quad (2.31)$$

It follows that (2.30) are 27 scalar equations for 27 unknowns. By computing the explicit solution of this algebraic system, we arrive at (2.27). \square

Remark 2.5. *Due to our symmetry, see Lemma 2.2, obtained in the above subsection we can replace the minors by conjugate terms among the representation of the functions $S_n(k)$. It may look like much simple to compute the jump matrices $J_{m,n}(x, t, k)$.*

2.8. The residue conditions. Since μ_2 is an entire function, it follows from (2.26) that M can only have singularities at the points where the S'_n s have singularities. We denote the possible zeros by $\{k_j\}_1^N$ and assume they satisfy the following assumption.

Assumption 2.6. *We assume that*

- $m_{11}(\mathcal{A})(k)$ has n_0 possible simple zeros in D_1 denoted by $\{k_j\}_1^{n_0}$;
- $(S^T s^A)_{11}(k)$ has $n_1 - n_0$ possible simple zeros in D_2 denoted by $\{k_j\}_{n_0+1}^{n_1}$;
- $(s^T S^A)_{11}(k)$ has $n_2 - n_1$ possible simple zeros in D_3 denoted by $\{k_j\}_{n_1+1}^{n_2}$;
- $\mathcal{A}_{11}(k)$ has $N - n_2$ possible simple zeros in D_4 denoted by $\{k_j\}_{n_2+1}^N$;

and that none of these zeros coincide. Moreover, we assume that none of these functions have zeros on the boundaries of the D_n 's.

We determine the residue conditions at these zeros in the following:

Proposition 2.7. *Let $\{M_n\}_1^4$ be the eigenfunctions defined by (2.9) and assume that the set $\{k_j\}_1^N$ of singularities are as the above assumption. Then the following residue conditions hold:*

$$Res_{k=k_j}[M]_1 = \frac{\mathcal{A}_{33}(k_j)[M(k_j)]_2 - \mathcal{A}_{23}(k_j)[M(k_j)]_3}{\dot{m}_{11}(\mathcal{A})(k_j)m_{21}(\mathcal{A})(k_j)} e^{2\theta(k_j)}, \quad 1 \leq j \leq n_0, k_j \in D_1 \quad (2.32a)$$

$$Res_{k=k_j}[M]_1 = \frac{S_{21}(k_j)s_{33}(k_j) - S_{31}(k_j)s_{23}(k_j)}{(S^T s^A)_{33}(k_j)m_{11}(k_j)} e^{2\theta(k_j)} [M(k_j)]_2 + \frac{S_{31}(k_j)s_{22}(k_j) - S_{21}(k_j)s_{32}(k_j)}{(S^T s^A)_{33}(k_j)m_{11}(k_j)} e^{2\theta(k_j)} [M(k_j)]_3 \quad (2.32b)$$

$$n_0 + 1 \leq j \leq n_1, k_j \in D_2,$$

$$Res_{k=k_j}[M]_2 = \frac{m_{33}(s)(k_j)M_{21}(S)(k_j) - m_{23}(s)(k_j)M_{31}(S)(k_j)}{(s^T S^A)_{11}(k_j)s_{11}(k_j)} e^{-2\theta(k_j)} [M(k_j)]_1 \quad (2.32c)$$

$$n_1 + 1 \leq j \leq n_2, k_j \in D_3,$$

$$Res_{k=k_j}[M]_3 = \frac{m_{32}(s)(k_j)M_{21}(S)(k_j) - m_{22}(s)(k_j)M_{31}(S)(k_j)}{(s^T S^A)_{11}(k_j)s_{11}(k_j)} e^{-2\theta(k_j)} [M(k_j)]_1 \quad (2.32d)$$

$$n_1 + 1 \leq j \leq n_2, k_j \in D_3.$$

$$Res_{k=k_j}[M]_2 = \frac{m_{33}(s)(k_j)}{\dot{s}_{11}(k_j)s_{21}(k_j)} e^{-2\theta(k_j)} [M(k_j)]_1, \quad n_2 + 1 \leq j \leq N, k_j \in D_4. \quad (2.32e)$$

$$Res_{k=k_j}[M]_3 = \frac{m_{32}(s)(k_j)}{\dot{s}_{11}(k_j)s_{21}(k_j)} e^{-2\theta(k_j)} [M(k_j)]_1, \quad n_2+1 \leq j \leq N, k_j \in D_4. \quad (2.32f)$$

where $\dot{f} = \frac{df}{dk}$, and θ is defined by

$$\theta(x, t, k) = ikx + 2ik^2t. \quad (2.33)$$

Proof. We will prove (2.32a), (2.32c), the other conditions follow by similar arguments. Equation (2.26) implies the relation

$$M_1 = \mu_2 e^{(ikx+2ik^2t)\hat{\Lambda}} S_1, \quad (2.34a)$$

$$M_3 = \mu_2 e^{(ikx+2ik^2t)\hat{\Lambda}} S_3, \quad (2.34b)$$

In view of the expressions for S_1 and S_3 given in (2.27), the three columns of (2.34a) read:

$$[M_1]_1 = [\mu_2]_1 \frac{1}{m_{11}(\mathcal{A})}, \quad (2.35a)$$

$$[M_1]_2 = [\mu_2]_1 e^{-2\theta} \mathcal{A}_{12} + [\mu_2]_2 \mathcal{A}_{22} + [\mu_2]_3 \mathcal{A}_{32}, \quad (2.35b)$$

$$[M_1]_3 = [\mu_2]_1 e^{-2\theta} \mathcal{A}_{13} + [\mu_2]_2 \mathcal{A}_{23} + [\mu_2]_3 \mathcal{A}_{33}. \quad (2.35c)$$

while the three columns of (2.34b) read:

$$[M_3]_1 = [\mu_2]_1 s_{11} + [\mu_2]_2 s_{21} e^{2\theta} + [\mu_2]_3 s_{31} e^{2\theta} \quad (2.36a)$$

$$\begin{aligned} [M_3]_2 &= [\mu_2]_1 \frac{m_{33}(s)m_{21}(S) - m_{23}(s)m_{31}(S)}{(s^T S A)_{11}} e^{-2\theta} \\ &\quad + [\mu_2]_2 \frac{m_{33}(s)m_{11}(S) - m_{13}(s)m_{31}(S)}{(s^T S A)_{11}} \\ &\quad + [\mu_2]_3 \frac{m_{23}(s)m_{11}(S) - m_{13}(s)m_{21}(S)}{(s^T S A)_{11}} \end{aligned} \quad (2.36b)$$

$$\begin{aligned} [M_3]_3 &= [\mu_2]_1 \frac{m_{32}(s)m_{21}(S) - m_{22}(s)m_{31}(S)}{(s^T S A)_{11}} e^{-2\theta} \\ &\quad + [\mu_2]_2 \frac{m_{32}(s)m_{11}(S) - m_{12}(s)m_{31}(S)}{(s^T S A)_{11}} \\ &\quad + [\mu_2]_3 \frac{m_{22}(s)m_{11}(S) - m_{12}(s)m_{21}(S)}{(s^T S A)_{11}}. \end{aligned} \quad (2.36c)$$

We first suppose that $k_j \in D_1$ is a simple zero of $m_{11}(\mathcal{A})(k)$. Solving (2.35b) and (2.35c) for $[\mu_2]_1, [\mu_2]_3$ and substituting the result in to (2.35a), we find

$$[M_1]_1 = \frac{\mathcal{A}_{33}[M_1]_2 - \mathcal{A}_{32}[M_1]_3}{m_{11}(\mathcal{A})m_{21}(\mathcal{A})} e^{2\theta} - \frac{[\mu_2]_2}{m_{21}(\mathcal{A})} e^{2\theta}.$$

Taking the residue of this equation at k_j , we find the condition (2.32a) in the case when $k_j \in D_1$.

In order to prove (2.32c), we solve (2.36a) for $[\mu_2]_1$, then substituting the result into (2.36b) and (2.36c), we find

$$[M_3]_2 = \frac{m_{33}(s)}{s_{11}}[\mu_2]_2 + \frac{m_{23}(s)}{s_{11}}[\mu_2]_3 + \frac{m_{33}(s)m_{21}(S) - m_{23}(s)m_{31}(S)}{(s^T \dot{S}^A)_{11}s_{11}}e^{-2\theta}[M_3]_1, \quad (2.37a)$$

$$[M_3]_3 = \frac{m_{32}(s)}{s_{11}}[\mu_2]_2 + \frac{m_{22}(s)}{s_{11}}[\mu_2]_3 + \frac{m_{32}(s)m_{21}(S) - m_{22}(s)m_{31}(S)}{(s^T \dot{S}^A)_{11}s_{11}}e^{-2\theta}[M_3]_1. \quad (2.37b)$$

Taking the residue of this equation at k_j , we find the condition (2.32c) in the case when $k_j \in D_3$. \square

2.9. The global relation. The spectral functions $S(k)$, $S_L(k)$ and $s(k)$ are not independent but satisfy an important relation. Indeed, it follows from (2.24) that

$$\mu_1(x, t, k)e^{(ikx+2ik^2t)\hat{\Lambda}}\{S^{-1}(k)s(k)e^{-ikL\hat{\Lambda}}S_L(k)\} = \mu_4(x, t, k). \quad (2.38)$$

Since $\mu_1(0, T, k) = \mathbb{I}$, evaluation at $(0, T)$ yields the following global relation:

$$S^{-1}(k)s(k)e^{-ikL\hat{\Lambda}}S_L(k) = e^{-2ik^2T\hat{\Lambda}}c(T, k), \quad (2.39)$$

where $c(T, k) = \mu_4(0, T, k)$.

3. THE RIEMANN-HILBERT PROBLEM

The sectionally analytic function $M(x, t, k)$ defined in section 2 satisfies a Riemann-Hilbert problem which can be formulated in terms of the initial and boundary values of $q_1(x, t)$ and $q_2(x, t)$. By solving this Riemann-Hilbert problem, the solution of (1.1) can be recovered for all values of x, t .

Theorem 3.1. *Suppose that $q_1(x, t)$ and $q_2(x, t)$ are a pair of solutions of (1.1) in the interval domain Ω . Then $q_1(x, t)$ and $q_2(x, t)$ can be reconstructed from the initial value $\{q_{10}(x), q_{20}(x)\}$ and boundary values*

$\{g_{01}(t), g_{02}(t), g_{11}(t), g_{12}(t)\}, \{f_{01}(t), f_{02}(t), f_{11}(t), f_{12}(t)\}$ defined as follows,

$$\begin{aligned} q_{10}(x) &= q_1(x, t=0), & q_{20}(x) &= q_2(x, t=0), \\ g_{01}(t) &= q_1(x=0, t), & g_{02}(t) &= q_2(x=0, t), \\ f_{01}(t) &= q_1(x=L, t), & f_{02}(t) &= q_2(x=L, t), \\ g_{11}(t) &= q_{1x}(x=0, t), & g_{12}(t) &= q_{2x}(x=0, t), \\ f_{11}(t) &= q_{1x}(x=L, t), & f_{12}(t) &= q_{2x}(x=L, t). \end{aligned} \quad (3.1)$$

Use the initial and boundary data to define the jump matrices $J_{m,n}(x, t, k)$ in terms of the spectral functions $s(k)$ and $S(k), S_L(k)$ by equation (2.24).

Assume that the possible zeros $\{k_j\}_1^N$ of the functions $m_{11}(\mathcal{A})(k)$, $(S^T s^A)_{11}(k)$, $(s^T S^A)_{11}(k)$ and $\mathcal{A}_{11}(k)$ are as in assumption 2.6.

Then the solution $\{q_1(x, t), q_2(x, t)\}$ is given by

$$q_1(x, t) = 2i \lim_{k \rightarrow \infty} (kM(x, t, k))_{12}, \quad q_2(x, t) = 2i \lim_{k \rightarrow \infty} (kM(x, t, k))_{13}. \quad (3.2)$$

where $M(x, t, k)$ satisfies the following 3×3 matrix Riemann-Hilbert problem:

- M is sectionally meromorphic on the Riemann k -sphere with jumps across the contours $\bar{D}_n \cap \bar{D}_m, n, m = 1, \dots, 4$, see Figure 2.
- Across the contours $\bar{D}_n \cap \bar{D}_m$, M satisfies the jump condition

$$M_n(x, t, k) = M_m(x, t, k) J_{m,n}(x, t, k), \quad k \in \bar{D}_n \cap \bar{D}_m, n, m = 1, 2, 3, 4. \quad (3.3)$$

- $M(x, t, k) = \mathbb{I} + O(\frac{1}{k})$, $k \rightarrow \infty$.
- The residue condition of M is showed in Proposition 2.7.

Proof. It only remains to prove (3.2) and this equation follows from the large k asymptotics of the eigenfunctions. \square

4. NON-LINEARIZABLE BOUNDARY CONDITIONS

A major difficulty of initial-boundary value problems is that some of the boundary values are unknown for a well-posed problem. All boundary values are needed for the definition of $S(k)$, $S_L(k)$, and hence for the formulation of the Riemann-Hilbert problem. Our main result expresses the unknown boundary data in terms of the prescribed boundary data and the initial data via the solution of a system of nonlinear integral equations.

4.1. Asymptotics. An analysis of (2.2) shows that the eigenfunctions $\{\mu_j\}_1^4$ have the following asymptotics as $k \rightarrow \infty$:

$$\begin{aligned}
\mu_j(x, t, k) &= \mathbb{I} + \frac{1}{k} \begin{pmatrix} \mu_{11}^{(1)} & \mu_{12}^{(1)} & \mu_{13}^{(1)} \\ \mu_{21}^{(1)} & \mu_{22}^{(1)} & \mu_{23}^{(1)} \\ \mu_{31}^{(1)} & \mu_{32}^{(1)} & \mu_{33}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \mu_{11}^{(2)} & \mu_{12}^{(2)} & \mu_{13}^{(2)} \\ \mu_{21}^{(2)} & \mu_{22}^{(2)} & \mu_{23}^{(2)} \\ \mu_{31}^{(2)} & \mu_{32}^{(2)} & \mu_{33}^{(2)} \end{pmatrix} + O(\frac{1}{k^3}) \\
&= \mathbb{I} + \frac{1}{k} \begin{pmatrix} \int_{(x_j, t_j)}^{(x, t)} \Delta_{11} & \frac{q_1}{2i} & \frac{q_2}{2i} \\ -\frac{\sigma \bar{q}_1}{2i} & \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)} & \int_{(x_j, t_j)}^{(x, t)} \Delta_{23}^{(1)} \\ -\frac{\sigma \bar{q}_2}{2i} & \int_{(x_j, t_j)}^{(x, t)} \Delta_{32}^{(1)} & \int_{(x_j, t_j)}^{(x, t)} \Delta_{33}^{(1)} \end{pmatrix} \\
&+ \frac{1}{k^2} \begin{pmatrix} \mu_{11}^{(2)} & \frac{q_1 \mu_{22}^{(1)} + q_2 \mu_{32}^{(1)}}{2i} + \frac{1}{4} q_{1x} & \frac{q_1 \mu_{23}^{(1)} + q_2 \mu_{33}^{(1)}}{2i} + \frac{1}{4} q_{2x} \\ \frac{1}{4} \sigma \bar{q}_{1x} - \frac{\sigma}{2i} \bar{q}_1 \mu_{11}^{(1)} & \mu_{22}^{(2)} & \mu_{23}^{(2)} \\ \frac{1}{4} \sigma \bar{q}_{2x} - \frac{\sigma}{2i} \bar{q}_2 \mu_{11}^{(1)} & \mu_{32}^{(2)} & \mu_{33}^{(2)} \end{pmatrix} + O(\frac{1}{k^3}).
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
\Delta_{11} &= \sigma[\frac{i}{2}(|q_1|^2 + |q_2|^2)dx + \frac{1}{2}(q_1 \bar{q}_{1x} - \bar{q}_1 q_{1x} + q_2 \bar{q}_{2x} - \bar{q}_2 q_{2x})dt] \\
\Delta_{22}^{(1)} &= \sigma[-\frac{i}{2}|q_1|^2 dx - \frac{1}{2}(\bar{q}_{1x} q_1 - \bar{q}_1 q_{1x})dt] \\
\Delta_{23}^{(1)} &= \sigma[-\frac{i}{2}\bar{q}_1 q_2 dx - \frac{1}{2}(\bar{q}_{1x} q_2 - \bar{q}_1 q_{2x})dt] \\
\Delta_{32}^{(1)} &= \sigma[-\frac{i}{2}\bar{q}_2 q_1 dx - \frac{1}{2}(\bar{q}_{2x} q_1 - \bar{q}_2 q_{1x})dt] \\
\Delta_{33}^{(1)} &= \sigma[-\frac{i}{2}|q_2|^2 dx - \frac{1}{2}(\bar{q}_{2x} q_2 - \bar{q}_2 q_{2x})dt].
\end{aligned} \tag{4.2}$$

The functions $\{\mu_{jl}^{(i)} = \mu_{jl}^{(i)}(x, t)\}_{j,l=1}^3, i = 1, 2$ are independent of k .

Remark 4.1. *Because we do not need the asymptotic expressions of the $\mu_{11}^{(2)}$ and $\{\mu_{ij}^{(2)}\}_{i,j=2}^3$ in the following analysis, we do not write down the explicit formulas for these functions.*

We define functions $\{\Phi_{ij}(t, k)\}_{i,j=1}^3$ and $\{\phi_{ij}(t, k)\}_{i,j=1}^3$ by:

$$\mu_2(0, t, k) = \begin{pmatrix} \Phi_{11}(t, k) & \Phi_{12}(t, k) & \Phi_{13}(t, k) \\ \Phi_{21}(t, k) & \Phi_{22}(t, k) & \Phi_{23}(t, k) \\ \Phi_{31}(t, k) & \Phi_{32}(t, k) & \Phi_{33}(t, k) \end{pmatrix}, \quad (4.3)$$

$$\mu_3(L, t, k) = \begin{pmatrix} \phi_{11}(t, k) & \phi_{12}(t, k) & \phi_{13}(t, k) \\ \phi_{21}(t, k) & \phi_{22}(t, k) & \phi_{23}(t, k) \\ \phi_{31}(t, k) & \phi_{32}(t, k) & \phi_{33}(t, k) \end{pmatrix}. \quad (4.4)$$

From the asymptotic of $\mu_j(x, t, k)$ in (4.1) we have

$$\begin{aligned} \mu_2(0, t, k) &= \mathbb{I} + \frac{1}{k} \begin{pmatrix} \Phi_{11}^{(1)}(t) & \Phi_{12}^{(1)}(t) & \Phi_{13}^{(1)}(t) \\ \Phi_{21}^{(1)}(t) & \Phi_{22}^{(1)}(t) & \Phi_{23}^{(1)}(t) \\ \Phi_{31}^{(1)}(t) & \Phi_{32}^{(1)}(t) & \Phi_{33}^{(1)}(t) \end{pmatrix} \\ &\quad + \frac{1}{k^2} \begin{pmatrix} \Phi_{11}^{(2)}(t) & \Phi_{12}^{(2)}(t) & \Phi_{13}^{(2)}(t) \\ \Phi_{21}^{(2)}(t) & \Phi_{22}^{(2)}(t) & \Phi_{23}^{(2)}(t) \\ \Phi_{31}^{(2)}(t) & \Phi_{32}^{(2)}(t) & \Phi_{33}^{(2)}(t) \end{pmatrix} + O\left(\frac{1}{k^3}\right). \end{aligned} \quad (4.5)$$

Recalling that the definition of the boundary data at $x = 0$, we have

$$\begin{aligned} \Phi_{12}^{(1)}(t) &= \frac{1}{2i}g_{01}(t), & \Phi_{12}^{(2)}(t) &= \frac{1}{4}g_{11}(t) + \frac{(g_{01}(t)\Phi_{22}^{(1)}(t) + g_{02}(t)\Phi_{32}^{(1)}(t))}{2i}, \\ \Phi_{13}^{(1)}(t) &= \frac{1}{2i}g_{02}(t), & \Phi_{13}^{(2)}(t) &= \frac{1}{4}g_{12}(t) + \frac{(g_{01}(t)\Phi_{23}^{(1)}(t) + g_{02}(t)\Phi_{33}^{(1)}(t))}{2i}, \\ \Phi_{22}^{(1)}(t) &= -\frac{\sigma}{2} \int_0^t (\bar{g}_{11}(t)g_{01}(t) - \bar{g}_{01}(t)g_{11}(t))dt, & \Phi_{23}^{(1)}(t) &= -\frac{\sigma}{2} \int_0^t (\bar{g}_{11}(t)g_{02}(t) - \bar{g}_{01}(t)g_{12}(t))dt, \\ \Phi_{32}^{(1)}(t) &= -\frac{\sigma}{2} \int_0^t (\bar{g}_{12}(t)g_{01}(t) - \bar{g}_{02}(t)g_{11}(t))dt, & \Phi_{33}^{(1)}(t) &= -\frac{\sigma}{2} \int_0^t (\bar{g}_{12}(t)g_{02}(t) - \bar{g}_{02}(t)g_{12}(t))dt. \end{aligned} \quad (4.6)$$

In particular, we find the following expressions for the boudary values at $x = 0$:

$$g_{01}(t) = 2i\Phi_{12}^{(1)}(t), \quad g_{02}(t) = 2i\Phi_{13}^{(1)}(t), \quad (4.7a)$$

$$\begin{aligned} g_{11}(t) &= 4\Phi_{12}^{(2)}(t) + 2i(g_{01}(t)\Phi_{22}^{(1)}(t) + g_{02}(t)\Phi_{32}^{(1)}(t)), \\ g_{12}(t) &= 4\Phi_{13}^{(2)}(t) + 2i(g_{01}(t)\Phi_{23}^{(1)}(t) + g_{02}(t)\Phi_{33}^{(1)}(t)) \end{aligned} \quad (4.7b)$$

Similarly, we have the asymptotic formulas for $\mu_3(L, t, k) = \{\phi_{ij}(t, k)\}_{i,j=1}^3$,

$$\begin{aligned} \mu_3(L, t, k) = & \mathbb{I} + \frac{1}{k} \begin{pmatrix} \phi_{11}^{(1)}(t) & \phi_{12}^{(1)}(t) & \phi_{13}^{(1)}(t) \\ \phi_{21}^{(1)}(t) & \phi_{22}^{(1)}(t) & \phi_{23}^{(1)}(t) \\ \phi_{31}^{(1)}(t) & \phi_{32}^{(1)}(t) & \phi_{33}^{(1)}(t) \end{pmatrix} \\ & + \frac{1}{k^2} \begin{pmatrix} \phi_{11}^{(2)}(t) & \phi_{12}^{(2)}(t) & \phi_{13}^{(2)}(t) \\ \phi_{21}^{(2)}(t) & \phi_{22}^{(2)}(t) & \phi_{23}^{(2)}(t) \\ \phi_{31}^{(2)}(t) & \phi_{32}^{(2)}(t) & \phi_{33}^{(2)}(t) \end{pmatrix} + O(\frac{1}{k^3}). \end{aligned} \quad (4.8)$$

Recalling that the definition of the boundary data at $x = L$, we have

$$\begin{aligned} \phi_{12}^{(1)}(t) &= \frac{1}{2i} f_{01}(t), & \phi_{12}^{(2)}(t) &= \frac{1}{4} f_{11}(t) + \frac{(f_{01}(t)\phi_{22}^{(1)}(t) + f_{02}(t)\phi_{32}^{(1)}(t))}{2i}, \\ \phi_{13}^{(1)}(t) &= \frac{1}{2i} f_{02}(t), & \phi_{13}^{(2)}(t) &= \frac{1}{4} f_{12}(t) + \frac{(f_{01}(t)\phi_{23}^{(1)}(t) + f_{02}(t)\phi_{33}^{(1)}(t))}{2i}, \\ \phi_{22}^{(1)}(t) &= -\frac{\sigma}{2} \int_0^t (\bar{f}_{11}(t)f_{01}(t) - \bar{f}_{01}(t)f_{11}(t))dt, & \phi_{23}^{(1)}(t) &= -\frac{\sigma}{2} \int_0^t (\bar{f}_{11}(t)f_{02}(t) - \bar{f}_{01}(t)f_{12}(t))dt, \\ \phi_{32}^{(1)}(t) &= -\frac{\sigma}{2} \int_0^t (\bar{f}_{12}(t)f_{01}(t) - \bar{f}_{02}(t)f_{11}(t))dt, & \phi_{33}^{(1)}(t) &= -\frac{\sigma}{2} \int_0^t (\bar{f}_{12}(t)f_{02}(t) - \bar{f}_{02}(t)f_{12}(t))dt. \end{aligned} \quad (4.9)$$

In particular, we find the following expressions for the boudary values at $x = L$:

$$f_{01}(t) = 2i\phi_{12}^{(1)}(t), \quad f_{02}(t) = 2i\phi_{13}^{(1)}(t), \quad (4.10a)$$

$$\begin{aligned} f_{11}(t) &= 4\phi_{12}^{(2)}(t) + 2i(f_{01}(t)\phi_{22}^{(1)}(t) + f_{02}(t)\phi_{32}^{(1)}(t)), \\ f_{12}(t) &= 4\phi_{13}^{(2)}(t) + 2i(f_{01}(t)\phi_{23}^{(1)}(t) + f_{02}(t)\phi_{33}^{(1)}(t)) \end{aligned} \quad (4.10b)$$

From the global relation (2.39) and replacing T by t , we find

$$\mu_2(0, t, k) e^{2ik^2 t \hat{\Lambda}} \{s(k) e^{-ikL \hat{\Lambda}} S_L(k)\} = c(t, k). \quad (4.11)$$

From the relation (2.25c) and the symmetry (2.20), we know that the spectral function $S_L(k)$ can be expressed by $\{\phi_{ij}(t, k)\}_{i,j=1}^3$. So if we denote the matrix-value function $c(t, k)$ as $c(t, k) = (c_{ij}(t, k))_{i,j=1}^3$. The functions $\{c_{ij}(t, k)\}_{i=1,j=2}^3$ are analytic and bounded in D_1 away from the possible zeros of $m_{11}(\mathcal{A})(k)$ and of order $O(\frac{1+e^{2ikL}}{k})$ as $k \rightarrow \infty$.

In the vanishing initial value case, the asymptotic of $c_{1j}(t, k)$, $j = 2, 3$ becomes much more simple.

Lemma 4.2. *We assuming that the initial value and boundary value are compatible at $x = 0$ and $x = L$ (i.e. at $x = 0$, $q_{10}(0) = g_{01}(0)$, $q_{20}(0) =$*

$g_{02}(0)$; at $x = L$, $q_{10}(L) = f_{01}(0)$, $q_{20}(L) = f_{02}(0)$). Then, in the vanishing initial value case, the global relation (4.11) implies that the large k behavior of $c_{1j}(t, k)$, $j = 2, 3$ satisfies

$$\begin{aligned} c_{12}(t, k) &= \frac{\Phi_{12}^{(1)}(t)}{k} + \frac{\Phi_{12}^{(2)}(t) + \Phi_{12}^{(1)}(t)\bar{\phi}_{22}^{(1)}(t) + \Phi_{13}^{(1)}(t)\bar{\Phi}_{23}^{(1)}(t)}{k^2} + O\left(\frac{1}{k^3}\right) \\ &\quad - \sigma \left[\frac{\bar{\phi}_{21}^{(1)}(t)}{k} + \frac{\bar{\phi}_{21}^{(2)}(t) + \Phi_{11}^{(1)}(t)\bar{\phi}_{21}^{(1)}(t)}{k^2} + O\left(\frac{1}{k^3}\right) \right] e^{2ikL} \quad k \rightarrow \infty, \end{aligned} \quad (4.12a)$$

$$\begin{aligned} c_{13}(t, k) &= \frac{\Phi_{13}^{(1)}(t)}{k} + \frac{\Phi_{13}^{(2)}(t) + \Phi_{12}^{(1)}(t)\bar{\phi}_{32}^{(1)}(t) + \Phi_{13}^{(1)}(t)\bar{\Phi}_{33}^{(1)}(t)}{k^2} + O\left(\frac{1}{k^3}\right) \\ &\quad - \sigma \left[\frac{\bar{\phi}_{31}^{(1)}(t)}{k} + \frac{\bar{\phi}_{31}^{(2)}(t) + \Phi_{11}^{(1)}(t)\bar{\phi}_{31}^{(1)}(t)}{k^2} + O\left(\frac{1}{k^3}\right) \right] e^{2ikL} \quad k \rightarrow \infty. \end{aligned} \quad (4.12b)$$

Proof. The global relation shows that under the assumption of vanishing initial value

$$c_{12}(t, k) = \Phi_{12}(t, k)\bar{\phi}_{22}(t, \bar{k}) + \Phi_{13}(t, k)\bar{\phi}_{23}(t, \bar{k}) - \sigma\Phi_{11}\bar{\phi}_{21}(t, \bar{k})e^{2ikL}, \quad (4.13a)$$

$$c_{13}(t, k) = \Phi_{12}(t, k)\bar{\phi}_{32}(t, \bar{k}) + \Phi_{13}(t, k)\bar{\phi}_{33}(t, \bar{k}) - \sigma\Phi_{11}\bar{\phi}_{31}(t, \bar{k})e^{2ikL} \quad (4.13b)$$

Recalling the equation

$$\mu_t - 2ik^2[\Lambda, \mu] = V_2\mu. \quad (4.14)$$

From the first column of the equation (4.14) we get

$$\begin{cases} \Phi_{11t} = 2k(g_{01}\Phi_{21} + g_{02}\Phi_{31}) - i\sigma(|g_{01}|^2 + |g_{02}|^2)\Phi_{11} + i(g_{11}\Phi_{21} + g_{12}\Phi_{31}), \\ \Phi_{21t} - 4ik^2\Phi_{21} = 2\sigma k\bar{g}_{01}\Phi_{11} + i\sigma(|g_{01}|^2\Phi_{21} + \bar{g}_{01}g_{02}\Phi_{31}) - i\sigma\bar{g}_{11}\Phi_{11}, \\ \Phi_{31t} - 4ik^2\Phi_{31} = 2\sigma k\bar{g}_{02}\Phi_{11} + i\sigma(\bar{g}_{02}g_{01}\Phi_{21} + |g_{02}|^2\Phi_{31}) - i\sigma\bar{g}_{12}\Phi_{11}, \end{cases} \quad (4.15a)$$

From the second column of the equation (4.14) we get

$$\begin{cases} \Phi_{12t} + 4ik^2\Phi_{12} = 2k(g_{01}\Phi_{22} + g_{02}\Phi_{32}) - i\sigma(|g_{01}|^2 + |g_{02}|^2)\Phi_{12} + i(g_{11}\Phi_{22} + g_{12}\Phi_{32}), \\ \Phi_{22t} = 2\sigma k\bar{g}_{01}\Phi_{12} + i\sigma(|g_{01}|^2\Phi_{22} + \bar{g}_{01}g_{02}\Phi_{32}) - i\sigma\bar{g}_{11}\Phi_{12}, \\ \Phi_{32t} = 2\sigma k\bar{g}_{02}\Phi_{12} + i\sigma(\bar{g}_{02}g_{01}\Phi_{22} + |g_{02}|^2\Phi_{32}) - i\sigma\bar{g}_{12}\Phi_{12}, \end{cases} \quad (4.15b)$$

From the third column of the equation (4.14) we get

$$\begin{cases} \Phi_{13t} + 4ik^2\Phi_{13} = 2k(g_{01}\Phi_{23} + g_{02}\Phi_{33}) - i\sigma(|g_{01}|^2 + |g_{02}|^2)\Phi_{13} + i(g_{11}\Phi_{23} + g_{12}\Phi_{33}), \\ \Phi_{23t} = 2\sigma k\bar{g}_{01}\Phi_{13} + i\sigma(|g_{01}|^2\Phi_{23} + \bar{g}_{01}g_{02}\Phi_{33}) - i\sigma\bar{g}_{11}\Phi_{13}, \\ \Phi_{33t} = 2\sigma k\bar{g}_{02}\Phi_{13} + i\sigma(\bar{g}_{02}g_{01}\Phi_{23} + |g_{02}|^2\Phi_{33}) - i\sigma\bar{g}_{12}\Phi_{13}, \end{cases} \quad (4.15c)$$

Suppose

$$\begin{pmatrix} \Phi_{11} \\ \Phi_{21} \\ \Phi_{31} \end{pmatrix} = \left(\alpha_0(t) + \frac{\alpha_1(t)}{k} + \frac{\alpha_2(t)}{k^2} + \cdots \right) + \left(\beta_0(t) + \frac{\beta_1(t)}{k} + \frac{\beta_2(t)}{k^2} + \cdots \right) e^{4ik^2t}, \quad (4.16)$$

where the coefficients $\alpha_j(t)$ and $\beta_j(t)$, $j = 0, 1, 2, \dots$, are independent of k and are 3×1 matrix functions.

To determine these coefficients, we substitute the above equation into equation (4.15a) and use the initial conditions

$$\alpha_0(0) + \beta_0(0) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T, \quad \alpha_1(0) + \beta_1(0) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T.$$

Then we get

$$\begin{aligned} \begin{pmatrix} \Phi_{11} \\ \Phi_{21} \\ \Phi_{31} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Phi_{11}^{(1)} \\ \Phi_{21}^{(1)} \\ \Phi_{31}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Phi_{11}^{(2)} \\ \Phi_{21}^{(2)} \\ \Phi_{31}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right) \\ &+ \left[\frac{1}{k} \begin{pmatrix} 0 \\ -\Phi_{21}^{(1)}(0) \\ -\Phi_{31}^{(1)}(0) \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right] e^{4ik^2t} \end{aligned} \quad (4.17)$$

Similarly, suppose

$$\begin{pmatrix} \Phi_{12} \\ \Phi_{22} \\ \Phi_{32} \end{pmatrix} = \left(\alpha_0(t) + \frac{\alpha_1(t)}{k} + \frac{\alpha_2(t)}{k^2} + \cdots \right) + \left(\beta_0(t) + \frac{\beta_1(t)}{k} + \frac{\beta_2(t)}{k^2} + \cdots \right) e^{-4ik^2t}, \quad (4.18)$$

where the coefficients $\alpha_j(t)$ and $\beta_j(t)$, $j = 0, 1, 2, \dots$, are independent of k and are 3×1 matrix functions.

To determine these coefficients, we substitute the above equation into equation (4.15b) and use the initial conditions

$$\alpha_0(0) + \beta_0(0) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, \quad \alpha_1(0) + \beta_1(0) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T.$$

Then we get

$$\begin{aligned} \begin{pmatrix} \Phi_{12} \\ \Phi_{22} \\ \Phi_{32} \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Phi_{12}^{(1)} \\ \Phi_{22}^{(1)} \\ \Phi_{32}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Phi_{12}^{(2)} \\ \Phi_{22}^{(2)} \\ \Phi_{32}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right) \\ &+ \left[\frac{1}{k} \begin{pmatrix} -\Phi_{12}^{(1)}(0) \\ 0 \\ 0 \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} -\Phi_{12}^{(2)}(0) + \Phi_{12}^{(1)}(0)\Phi_{22}^{(1)} + \Phi_{12}^{(1)}(0)\Phi_{32}^{(1)} \\ -\frac{i\sigma}{2}\bar{g}_{01}\Phi_{12}^{(1)}(0) \\ -\frac{i\sigma}{2}\bar{g}_{02}\Phi_{12}^{(1)}(0) \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right] e^{-4ik^2t} \end{aligned} \quad (4.19)$$

Similar to the derivation of $\Phi_{i2}, i = 1, 2, 3$, from (4.15c) we can get the asymptotic formulas of $\Phi_{i3}, i = 1, 2, 3$

$$\begin{aligned} \begin{pmatrix} \Phi_{13} \\ \Phi_{23} \\ \Phi_{33} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Phi_{13}^{(1)} \\ \Phi_{23}^{(1)} \\ \Phi_{33}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Phi_{13}^{(2)} \\ \Phi_{23}^{(2)} \\ \Phi_{33}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right) \\ &+ \left[\frac{1}{k} \begin{pmatrix} -\Phi_{13}^{(1)}(0) \\ 0 \\ 0 \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} -\Phi_{13}^{(2)}(0) + \Phi_{13}^{(1)}(0)\Phi_{23}^{(1)} + \Phi_{13}^{(1)}(0)\Phi_{33}^{(1)} \\ -\frac{i\sigma}{2}\bar{g}_{01}\Phi_{13}^{(1)}(0) \\ -\frac{i\sigma}{2}\bar{g}_{02}\Phi_{13}^{(1)}(0) \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right] e^{-4ik^2t} \end{aligned} \quad (4.20)$$

Similar to (4.15), we have the equations $\{\phi_{ij}\}_{i,j=1}^3$ satisfy the similar partial derivative equations:

From the first column of the equation (4.14) we get

$$\begin{cases} \phi_{11t} = 2k(f_{01}\phi_{21} + f_{02}\phi_{31}) - i\sigma(|f_{01}|^2 + |f_{02}|^2)\phi_{11} + i(f_{11}\phi_{21} + f_{12}\phi_{31}), \\ \phi_{21t} - 4ik^2\phi_{21} = 2\sigma k\bar{f}_{01}\phi_{11} + i\sigma(|f_{01}|^2\phi_{21} + \bar{f}_{01}f_{02}\phi_{31}) - i\sigma\bar{f}_{11}\phi_{11}, \\ \phi_{31t} - 4ik^2\phi_{31} = 2\sigma k\bar{f}_{02}\phi_{11} + i\sigma(\bar{f}_{02}f_{01}\phi_{21} + |f_{02}|^2\phi_{31}) - i\sigma\bar{f}_{12}\phi_{11}, \end{cases} \quad (4.21a)$$

From the second column of the equation (4.14) we get

$$\begin{cases} \phi_{12t} + 4ik^2\phi_{12} = 2k(f_{01}\phi_{22} + f_{02}\phi_{32}) - i\sigma(|f_{01}|^2 + |f_{02}|^2)\phi_{12} + i(f_{11}\phi_{22} + f_{12}\phi_{32}), \\ \phi_{22t} = 2\sigma k\bar{f}_{01}\phi_{12} + i\sigma(|f_{01}|^2\phi_{22} + \bar{f}_{01}f_{02}\phi_{32}) - i\sigma\bar{f}_{11}\phi_{12}, \\ \phi_{32t} = 2\sigma k\bar{f}_{02}\phi_{12} + i\sigma(\bar{f}_{02}f_{01}\phi_{22} + |f_{02}|^2\phi_{32}) - i\sigma\bar{f}_{12}\phi_{12}, \end{cases} \quad (4.21b)$$

From the third column of the equation (4.14) we get

$$\begin{cases} \phi_{13t} + 4ik^2\phi_{13} = 2k(f_{01}\phi_{23} + f_{02}\phi_{33}) - i\sigma(|f_{01}|^2 + |f_{02}|^2)\phi_{13} + i(f_{11}\phi_{23} + f_{12}\phi_{33}), \\ \phi_{23t} = 2\sigma k\bar{f}_{01}\phi_{13} + i\sigma(|f_{01}|^2\phi_{23} + \bar{f}_{01}f_{02}\phi_{33}) - i\sigma\bar{f}_{11}\phi_{13}, \\ \phi_{33t} = 2\sigma k\bar{f}_{02}\phi_{13} + i\sigma(\bar{f}_{02}f_{01}\phi_{23} + |f_{02}|^2\phi_{33}) - i\sigma\bar{f}_{12}\phi_{13}, \end{cases} \quad (4.21c)$$

Then, substituting these formulas into the equation (4.13a) and noticing that we assume that the initial value and boundary value are compatible at $x = 0$ and $x = L$, we get the asymptotic behavior (4.12a) of $c_{1j}(t, k)$ as $k \rightarrow \infty$. Similar to prove the formula (4.12b). \square

4.2. The Dirichlet and Neumann problems. We can now derive effective characterizations of spectral function $S(k)$, $S_L(k)$ for the Dirichlet ($\{g_{01}(t), g_{02}(t)\}$ and $\{f_{01}(t), f_{02}(t)\}$ prescribed), the Neumann ($\{g_{11}(t), g_{12}(t)\}$ and $\{f_{11}(t), f_{12}(t)\}$ prescribed) problems.

Define functions as

$$f_-(t, k) = f(t, k) - f(t, -k), \quad f_+(t, k) = f(t, k) + f(t, -k), \quad (4.22)$$

Introducing

$$\Delta(k) = e^{2ikL} - e^{-2ikL}, \quad \Sigma(k) = e^{2ikL} + e^{-2ikL} \quad (4.23)$$

Denoting ∂D_3^0 as the boundary contour which is not included the zeros of $\Delta(k)$.

Theorem 4.3. *Let $T < \infty$. Let $q_{10}(x), q_{20}(x), 0 \leq x \leq L$, be two initial functions.*

For the Dirichlet problem it is assumed that the function $\{g_{01}(t), g_{02}(t)\}, 0 \leq t < T$, has sufficient smoothness and is compatible with $q_{10}(x), q_{20}(x)$

at $x = t = 0$, that is

$$q_{10}(0) = g_{01}(0), \quad q_{20}(0) = g_{02}(0).$$

The function $\{f_{01}(t), f_{02}(t)\}, 0 \leq t < T$, has sufficient smoothness and is compatible with $q_{10}(x), q_{20}(x)$ at $x = L$, that is,

$$q_{10}(L) = f_{01}(0), \quad q_{20}(L) = f_{02}(0).$$

For the Neumann problem it is assumed that the function $g_1(t), 0 \leq t < T$, has sufficient smoothness and is compatible with $q_0(x)$ at $x = t = 0$; the function $f_1(t), 0 \leq t < T$, has sufficient smoothness and is compatible with $q_0(x)$ at $x = L$.

For simplicity, we suppose that $m_{11}(\mathcal{A})(k)$ has no zeros in D_1 .

Then the spectral function $S(k)$ is given by

$$S(k) = \begin{pmatrix} \overline{\Phi_{11}(\bar{k})} & -\sigma \overline{\Phi_{21}(\bar{k})} e^{4ik^2 T} & -\sigma \overline{\Phi_{31}(\bar{k})} e^{4ik^2 T} \\ -\sigma \overline{\Phi_{12}(\bar{k})} e^{-4ik^2 T} & \overline{\Phi_{22}(\bar{k})} & \overline{\Phi_{32}(\bar{k})} \\ -\sigma \overline{\Phi_{13}(\bar{k})} e^{-4ik^2 T} & \overline{\Phi_{23}(\bar{k})} & \overline{\Phi_{33}(\bar{k})} \end{pmatrix} \quad (4.24)$$

$$S_L(k) = \begin{pmatrix} \overline{\phi_{11}(\bar{k})} & -\sigma \overline{\phi_{21}(\bar{k})} e^{4ik^2 T} & -\sigma \overline{\phi_{31}(\bar{k})} e^{4ik^2 T} \\ -\sigma \overline{\phi_{12}(\bar{k})} e^{-4ik^2 T} & \overline{\phi_{22}(\bar{k})} & \overline{\phi_{32}(\bar{k})} \\ -\sigma \overline{\phi_{13}(\bar{k})} e^{-4ik^2 T} & \overline{\phi_{23}(\bar{k})} & \overline{\phi_{33}(\bar{k})} \end{pmatrix} \quad (4.25)$$

and the complex-value functions $\{\Phi_{l3}(t, k)\}_{l=1}^3$ satisfy the following system of integral equations:

$$\begin{aligned} \Phi_{13}(t, k) &= \int_0^t e^{-4ik^2(t-t')} [-i\sigma(|g_{01}|^2 + |g_{02}|^2)\Phi_{13} \\ &\quad + (2kg_{01} + ig_{11})\Phi_{23} + (2kg_{02} + ig_{12})\Phi_{33}] (t', k) dt' \end{aligned} \quad (4.26a)$$

$$\Phi_{23}(t, k) = \int_0^t \sigma [(2k\bar{g}_{01} - i\bar{g}_{11})\Phi_{13} + i|g_{01}|^2\Phi_{23} + i\bar{g}_{01}g_{02}\Phi_{33}] (t', k) dt' \quad (4.26b)$$

$$\Phi_{33}(t, k) = 1 + \int_0^t \sigma [(2k\bar{g}_{02} - i\bar{g}_{12})\Phi_{13} + ig_{01}\bar{g}_{02}\Phi_{23} + i|g_{02}|^2\Phi_{33}] (t', k) dt' \quad (4.26c)$$

and $\{\Phi_{l1}(t, k)\}_{l=1}^3, \{\Phi_{l2}(t, k)\}_{l=1}^3$ satisfy the following system of integral equations:

$$\begin{aligned} \Phi_{11}(t, k) &= 1 + \int_0^t [-i\sigma(|g_{01}|^2 + |g_{02}|^2)\Phi_{11} \\ &\quad + (2kg_{01} + ig_{11})\Phi_{21} + (2kg_{02} + ig_{12})\Phi_{31}] (t', k) dt' \end{aligned} \quad (4.27a)$$

$$\Phi_{21}(t, k) = \int_0^t e^{4ik^2(t-t')} \sigma [(2k\bar{g}_{01} - i\bar{g}_{11})\Phi_{11} + i|g_{01}|^2\Phi_{21} + i\bar{g}_{01}g_{02}\Phi_{31}] (t', k) dt' \quad (4.27b)$$

$$\Phi_{31}(t, k) = \int_0^t e^{4ik^2(t-t')} \sigma [(2k\bar{g}_{02} - i\bar{g}_{12})\Phi_{11} + ig_{01}\bar{g}_{02}\Phi_{21} + i|g_{02}|^2\Phi_{31}] (t', k) dt' \quad (4.27c)$$

$$\begin{aligned} \Phi_{12}(t, k) &= \int_0^t e^{-4ik^2(t-t')} [-i\sigma(|g_{01}|^2 + |g_{02}|^2)\Phi_{12} \\ &\quad + (2kg_{01} + ig_{11})\Phi_{22} + (2kg_{02} + ig_{12})\Phi_{32}] (t', k) dt' \end{aligned} \quad (4.28a)$$

$$\Phi_{22}(t, k) = 1 + \int_0^t \sigma [(2k\bar{g}_{01} - i\bar{g}_{11})\Phi_{12} + i|g_{01}|^2\Phi_{22} + i\bar{g}_{01}g_{02}\Phi_{32}] (t', k) dt' \quad (4.28b)$$

$$\Phi_{32}(t, k) = \int_0^t \sigma [(2k\bar{g}_{02} - i\bar{g}_{12})\Phi_{12} + ig_{01}\bar{g}_{02}\Phi_{22} + i|g_{02}|^2\Phi_{32}] (t', k) dt' \quad (4.28c)$$

Functions $\{\phi_{ij}(t, k)\}_{i,j=1}^3$ satisfy the same integral equations replaying the functions $\{g_{01}, g_{02}, g_{11}, g_{12}\}$ with $\{f_{01}, f_{02}, f_{11}, f_{12}\}$.

(i) For the Dirichlet problem, the unknown Neumann boundary value $\{g_{11}(t), g_{12}(t)\}$ and $\{f_{11}(t), f_{12}(t)\}$ are given by

$$\begin{aligned} g_{11}(t) &= \frac{2}{i\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta} (k\Phi_{12-} + ig_{01}) dk + \frac{2}{\pi} \int_{\partial D_3^0} (g_{01}\Phi_{22-} + g_{02}\Phi_{32-}) dk \\ &\quad - \frac{1}{\pi} \int_{\partial D_3^0} (g_{01}\bar{\phi}_{22-} + g_{02}\bar{\phi}_{23-}) dk - \frac{4\sigma}{i\pi} \int_{\partial D_3^0} \frac{1}{\Delta} (k\bar{\phi}_{21-} + i\sigma\bar{f}_{01}) dk \\ &\quad + \frac{4}{i\pi} \int_{\partial D_3^0} \frac{k}{\Delta} [(\Phi_{12}(\bar{\phi}_{22} - 1) + \Phi_{13}\bar{\phi}_{23}) e^{-2ikL} + \sigma(\Phi_{11} - 1)\bar{\phi}_{21}]_- dk. \end{aligned} \quad (4.29a)$$

$$\begin{aligned} g_{12}(t) &= \frac{2}{i\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta} (k\Phi_{13-} + ig_{02}) dk + \frac{2}{\pi} \int_{\partial D_3^0} (g_{01}\Phi_{23-} + g_{02}\Phi_{33-}) dk \\ &\quad - \frac{1}{\pi} \int_{\partial D_3^0} (g_{01}\bar{\phi}_{32-} + g_{02}\bar{\phi}_{33-}) dk - \frac{4\sigma}{i\pi} \int_{\partial D_3^0} \frac{1}{\Delta} (k\bar{\phi}_{31-} + i\sigma\bar{f}_{02}) dk \\ &\quad + \frac{4}{i\pi} \int_{\partial D_3^0} \frac{k}{\Delta} [(\Phi_{12}\bar{\phi}_{32} + \Phi_{13}(\bar{\phi}_{33} - 1)) e^{-2ikL} + \sigma(\Phi_{11} - 1)\bar{\phi}_{31}]_- dk. \end{aligned} \quad (4.29b)$$

and

$$\begin{aligned}
f_{11}(t) &= -\frac{2}{i\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta} (k\phi_{12-} + if_{01}) dk + \frac{2}{\pi} \int_{\partial D_3^0} (f_{01}\phi_{22-} + f_{02}\phi_{32-}) dk \\
&+ \frac{1}{\pi} \int_{\partial D_3^0} (f_{01}\bar{\Phi}_{22-} + f_{02}\bar{\Phi}_{23-}) dk + \frac{4\sigma}{i\pi} \int_{\partial D_3^0} \frac{1}{\Delta} (k\bar{\Phi}_{21-} + i\sigma\bar{g}_{01}) dk \\
&+ \frac{4}{i\pi} \int_{\partial D_3^0} \frac{k}{\Delta} [\sigma(\phi_{11} - 1)\bar{\Phi}_{21} - (\phi_{12}(\bar{\Phi}_{22} - 1) + \phi_{13}\bar{\Phi}_{23}) e^{2ikL}]_- dk.
\end{aligned} \tag{4.30a}$$

$$\begin{aligned}
f_{12}(t) &= -\frac{2}{i\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta} (k\phi_{13-} + if_{02}) dk + \frac{2}{\pi} \int_{\partial D_3^0} (f_{03}\phi_{23-} + f_{02}\phi_{33-}) dk \\
&+ \frac{1}{\pi} \int_{\partial D_3^0} (f_{01}\bar{\Phi}_{32-} + f_{02}\bar{\Phi}_{33-}) dk + \frac{4\sigma}{i\pi} \int_{\partial D_3^0} \frac{1}{\Delta} (k\bar{\Phi}_{31-} + i\sigma\bar{g}_{02}) dk \\
&+ \frac{4}{i\pi} \int_{\partial D_3^0} \frac{k}{\Delta} [\sigma(\phi_{11} - 1)\bar{\Phi}_{31} - (\phi_{12}\bar{\Phi}_{32} + \phi_{13}(\bar{\Phi}_{33} - 1)) e^{2ikL}]_- dk.
\end{aligned} \tag{4.30b}$$

where the conjugate of a function h denotes $\bar{h} = \bar{h}(\bar{k})$.

- (ii) For the Neumann problem, the unknown boundary values $\{g_{01}(t), g_{02}(t)\}$ and $\{f_{01}(t), f_{02}(t)\}$ are given by

$$\begin{aligned}
g_{01}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta} \Phi_{12+} dk - \frac{2}{\pi} \int_{\partial D_3^0} \frac{\sigma}{\Delta} \bar{\Phi}_{21+} dk \\
&- \frac{2}{\pi} \int_{\partial D_3^0} \frac{1}{\Delta} [\sigma(\Phi_{11} - 1)\bar{\Phi}_{21} - (\Phi_{12}(\bar{\Phi}_{22} - 1) + \Phi_{13}\bar{\Phi}_{23}) e^{-2ikL}]_+ dk,
\end{aligned} \tag{4.31a}$$

$$\begin{aligned}
g_{02}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta} \Phi_{13+} dk - \frac{2}{\pi} \int_{\partial D_3^0} \frac{\sigma}{\Delta} \bar{\Phi}_{31+} dk \\
&- \frac{2}{\pi} \int_{\partial D_3^0} \frac{1}{\Delta} [\sigma(\Phi_{11} - 1)\bar{\Phi}_{31} - (\Phi_{12}\bar{\Phi}_{32} + \Phi_{13}(\bar{\Phi}_{33} - 1)) e^{-2ikL}]_+ dk,
\end{aligned} \tag{4.31b}$$

and

$$\begin{aligned}
f_{01}(t) &= -\frac{1}{\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta} \phi_{12+} dk - \frac{2}{\pi} \int_{\partial D_3^0} \frac{\sigma}{\Delta} \bar{\Phi}_{21+} dk \\
&- \frac{2}{\pi} \int_{\partial D_3^0} \frac{1}{\Delta} [\sigma(\phi_{11} - 1)\bar{\Phi}_{21} - (\phi_{12}(\bar{\Phi}_{22} - 1) + \phi_{13}\bar{\Phi}_{23}) e^{2ikL}]_+ dk,
\end{aligned} \tag{4.32a}$$

$$\begin{aligned}
f_{02}(t) &= -\frac{1}{\pi} \int_{\partial D_3^0} \frac{\Sigma}{\Delta} \phi_{13+} dk - \frac{2}{\pi} \int_{\partial D_3^0} \frac{\sigma}{\Delta} \bar{\Phi}_{31+} dk \\
&- \frac{2}{\pi} \int_{\partial D_3^0} \frac{1}{\Delta} [\sigma(\phi_{11} - 1)\bar{\Phi}_{31} - (\phi_{12}\bar{\Phi}_{32} + \phi_{13}(\bar{\Phi}_{33} - 1)) e^{2ikL}]_+ dk,
\end{aligned} \tag{4.32b}$$

Proof. The representations (4.24) and (4.25) follow from the relation $S(k) = e^{-2ik^2t\hat{\Lambda}}\mu_2^{-1}(0, t, k)$ and $S_L(k) = e^{-2ik^2t\hat{\Lambda}}\mu_3^{-1}(0, t, k)$, respectively. And the system (??) is the direct result of the Volteral integral equations of $\mu_2(0, t, k)$ and $\mu_3(L, t, k)$.

(i) In order to derive (4.29a) we note that equation (4.7b) expresses $g_{11}(t)$ in terms of $\Phi_{22}^{(1)}(t, k)$, $\Phi_{32}^{(1)}(t, k)$, $\Phi_{12}^{(2)}(t, k)$. Furthermore, equation (4.5) and Cauchy theorem imply

$$\begin{aligned} -\frac{\pi i}{2}\Phi_{22}^{(1)}(t) &= \int_{\partial D_2} [\Phi_{22}(t, k) - 1] dk = \int_{\partial D_4} [\Phi_{2 \times 2}(t, k) - 1] dk, \\ -\frac{\pi i}{2}\Phi_{32}^{(1)}(t) &= \int_{\partial D_2} \Phi_{32}(t, k) dk = \int_{\partial D_4} \Phi_{32}(t, k) dk, \end{aligned}$$

and

$$-\frac{\pi i}{2}\Phi_{12}^{(2)}(t) = \int_{\partial D_2} \left[k\Phi_{12}(t, k) - \frac{g_{01}(t)}{2i} \right] dk = \int_{\partial D_4} \left[k\Phi_{12}(t, k) - \frac{g_{01}(t)}{2i} \right] dk,$$

Thus,

$$\begin{aligned} i\pi\Phi_{22}^{(1)}(t) &= - \left(\int_{\partial D_2} + \int_{\partial D_4} \right) [\Phi_{22}(t, k) - 1] dk \\ &= \left(\int_{\partial D_1} + \int_{\partial D_3} \right) [\Phi_{22}(t, k) - 1] dk \\ &= \int_{\partial D_3} [\Phi_{22}(t, k) - 1] dk - \int_{\partial D_3} [\Phi_{22}(t, -k) - 1] dk \\ &= \int_{\partial D_3} (\Phi_{22}(t, k) - \Phi_{22}(t, -k)) dk \\ &= \int_{\partial D_3} \Phi_{22-}(t, k) dk \\ i\pi\Phi_{32}^{(1)}(t) &= - \left(\int_{\partial D_2} + \int_{\partial D_4} \right) [\Phi_{32}(t, k)] dk \\ &= \int_{\partial D_3} \Phi_{32-}(t, k) dk \end{aligned} \tag{4.33}$$

Similarly,

$$\begin{aligned} i\pi\Phi_{12}^{(2)}(t) &= \left(\int_{\partial D_3} + \int_{\partial D_1} \right) \left[k\Phi_{12}(t, k) - \frac{g_{01}(t)}{2i} \right] dk \\ &= \int_{\partial D_3} \left[k\Phi_{12}(t, k) - \frac{g_{01}(t)}{2i} \right]_- dk \\ &= \int_{\partial D_3^0} \left\{ k\Phi_{12}(t, k) - \frac{g_{01}(t)}{2i} + \frac{2e^{-2ikL}}{\Delta} \left[k\Phi_{12}(t, k) - \frac{g_{01}(t)}{2i} \right] \right\}_- dk + I(t) \end{aligned} \tag{4.34}$$

where $I(t)$ is defined by

$$I(t) = - \int_{\partial D_3^0} \left\{ \frac{2e^{-2ikL}}{\Delta} \left[k\Phi_{12}(t, k) - \frac{g_{01}(t)}{2i} \right] \right\}_- dk$$

The last step involves using the global relation (??) to compute $I(t)$. That is,

$$\begin{aligned}
I(t) &= \int_{\partial D_3^0} \left\{ -\frac{2e^{-2ikL}}{\Delta} \left[kc_{12} - \Phi_{12}^{(1)} - \frac{\Phi_{12}^{(1)}\bar{\phi}_{22}^{(1)} + \Phi_{13}^{(1)}\bar{\phi}_{23}^{(1)}}{k} + \sigma\bar{\phi}_{21}^{(1)}e^{2ikL} \right] \right\}_- dk \\
&+ \int_{\partial D_3^0} \left\{ -\frac{2e^{-2ikL}}{\Delta} \left[\frac{\Phi_{12}^{(1)}\bar{\phi}_{22}^{(1)} + \Phi_{13}^{(1)}\bar{\phi}_{23}^{(1)}}{k} + \sigma(k\bar{\phi}_{21} - \bar{\phi}_{21}^{(1)})e^{2ikL} \right] \right\}_- dk \\
&+ \int_{\partial D_3^0} \left\{ \frac{2e^{-2ikL}}{\Delta} [k(\Phi_{12}(\bar{\phi}_{22} - 1) + \Phi_{13}\bar{\phi}_{23} + \sigma(\Phi_{11} - 1)\bar{\phi}_{21}e^{2ikL})] \right\}_- dk
\end{aligned} \tag{4.35}$$

Using the asymptotic (??) and Cauchy theorem to compute these terms on the right-hand side of equation (4.35), we find

$$\begin{aligned}
I(t) &= -i\pi\Phi_{12}^{(2)} - \int_{\partial D_3^0} \left[\frac{g_{01}}{2i}\bar{\phi}_{22-} - \frac{2\sigma}{\Delta}(k\bar{\phi}_{21-} + i\sigma f_{01}) \right] dk \\
&+ \int_{\partial D_3^0} \frac{2k}{\Delta} [(\Phi_{12}(\bar{\phi}_{22} - 1) + \Phi_{13}\bar{\phi}_{23})e^{-2ikL} + \sigma(\Phi_{11} - 1)\bar{\phi}_{21}]_- dk.
\end{aligned} \tag{4.36}$$

Equations (4.33), (4.34) and (4.36) together with (4.7b) yield (4.29a). Similarly, we can prove (4.29b).

The expression (4.30a) for $f_{11}(t)$ can be derived in a similar way. Indeed, we note that equation (4.10b) expresses $f_{11}(t)$ in terms of $\phi_{22}^{(1)}(t, k)$, $\phi_{32}^{(1)}(t, k)$, $\phi_{12}^{(2)}(t, k)$. These three functions satisfy the analog of equations (4.33) and 4.34. In particular, $\phi_{12}^{(2)}$ satisfies

$$\begin{aligned}
i\pi\phi_{12}^{(2)}(t) &= \int_{\partial D_3} \left[k\phi_{12} - \phi_{12}^{(1)} \right]_- dk \\
&= \int_{\partial D_3^0} \left\{ -\frac{\Sigma}{\Delta}(k\phi_{12} - \phi_{12}^{(1)}) \right\}_- dk + J(t)
\end{aligned} \tag{4.37}$$

where

$$J(t) = \int_{\partial D_3^0} \left\{ \frac{2e^{2ikL}}{\Delta}(k\phi_{12} - \phi_{12}^{(1)}) \right\}_- dk \tag{4.38}$$

Then using the global relation (??) to compute $J(t)$:

$$\begin{aligned}
J(t) &= \int_{\partial D_3^0} \left\{ -\frac{2}{\Delta} \left[k\sigma\bar{c}_{21} + \phi_{12}^{(1)}e^{2ikL} - \sigma\bar{\Phi}_{21}^{(1)} + \frac{\phi_{12}^{(1)}\bar{\Phi}_{22} + \phi_{13}\bar{\Phi}_{23}}{k}e^{2ikL} \right] \right\}_- dk \\
&+ \int_{\partial D_3^0} \frac{2}{\Delta} \left\{ \frac{\phi_{12}^{(1)}\bar{\Phi}_{22} + \phi_{13}\bar{\Phi}_{23}}{k}e^{2ikL} + \sigma(k\bar{\Phi}_{21} - \bar{\Phi}_{21}^{(1)}) \right\}_- dk \\
&+ \int_{\partial D_3^0} \frac{2k}{\Delta} [\sigma(\phi_{11} - 1)\bar{\Phi}_{21} - (\phi_{12}(\bar{\Phi}_{22} - 1) + \phi_{13}\bar{\phi}_{23})e^{2ikL}]_- dk
\end{aligned} \tag{4.39}$$

The equations (4.39), (4.37) together with the asymptotics of $c_{21}(t, k)$ yield (4.30a). The proof of (4.30b) is similar.

- (ii) In order to derive the representations (4.31a) relevant for the Neumann problem, we note that equation (4.7a) expresses g_{01} and g_{02} in terms of $\Phi_{12}^{(1)}$ and $\Phi_{13}^{(1)}$, respectively. Furthermore, equation (4.5) and Cauchy theorem imply

$$-\frac{\pi i}{2}\Phi_{12}^{(1)}(t) = \int_{\partial D_2} \Phi_{12}(t, k)dk = \int_{\partial D_4} \Phi_{12}(t, k)dk, \quad (4.40)$$

Thus,

$$\begin{aligned} i\pi\Phi_{12}^{(1)}(t) &= \left(\int_{\partial D_3} + \int_{\partial D_1} \right) \Phi_{12}(t, k)dk \\ &= \int_{\partial D_3} \Phi_{12-}(t, k)dk \\ &= \int_{\partial D_3^0} \left(\frac{\Sigma}{\Delta} \Phi_{12+}(t, k) \right) dk + K(t), \end{aligned} \quad (4.41)$$

where

$$K(t) = - \int_{\partial D_3^0} \frac{2}{\Delta} (e^{-2ikL} \Phi_{1j})_+ dk \quad (4.42)$$

and using the global relation and the asymptotic formulas of c_{12} , we have

$$K(t) = -i\pi\Phi_{12}^{(1)} - \int_{\partial D_3^0} \left\{ \frac{2\sigma}{\Delta} \bar{\phi}_{21+} + \frac{2}{\Delta} [\sigma(\Phi_{11} - 1)\bar{\phi}_{21} - (\Phi_{12}(\bar{\phi}_{22} - 1) + \Phi_{13}\bar{\phi}_{23}e^{-2ikL})]_+ \right\} dk \quad (4.43)$$

Equations (4.7a), (4.41) and (4.43) yields (4.31a). The proof of the other formulas is similar.

□

4.3. Effective characterizations. Substituting into the system (??) the expressions

$$\Phi_{ij} = \Phi_{ij,0} + \varepsilon\Phi_{ij,1} + \varepsilon^2\Phi_{ij,2} + \cdots, \quad i, j = 1, 2, 3. \quad (4.44a)$$

$$\phi_{ij} = \phi_{ij,0} + \varepsilon\phi_{ij,1} + \varepsilon^2\phi_{ij,2} + \cdots, \quad i, j = 1, 2, 3. \quad (4.44b)$$

$$g_{01} = \varepsilon g_{01}^{(1)} + \varepsilon^2 g_{01}^{(2)} + \cdots, \quad g_{02} = \varepsilon g_{02}^{(1)} + \varepsilon^2 g_{02}^{(2)} + \cdots, \quad (4.44c)$$

$$f_{01} = \varepsilon f_{01}^{(1)} + \varepsilon^2 f_{01}^{(2)} + \cdots, \quad f_{02} = \varepsilon f_{02}^{(1)} + \varepsilon^2 f_{02}^{(2)} + \cdots, \quad (4.44d)$$

$$g_{11} = \varepsilon g_{11}^{(1)} + \varepsilon^2 g_{11}^{(2)} + \cdots, \quad g_{12} = \varepsilon g_{12}^{(1)} + \varepsilon^2 g_{12}^{(2)} + \cdots, \quad (4.44e)$$

$$f_{11} = \varepsilon f_{11}^{(1)} + \varepsilon^2 f_{11}^{(2)} + \cdots, \quad f_{12} = \varepsilon f_{12}^{(1)} + \varepsilon^2 f_{12}^{(2)} + \cdots, \quad (4.44f)$$

where $\varepsilon > 0$ is a small parameter, we find that the terms of $O(1)$ give

$$O(1) : \begin{cases} \Phi_{13,0} = 0 & \Phi_{23,0} = 0 & \Phi_{33,0} = 1, \\ \Phi_{11,0} = 1 & \Phi_{21,0} = 0 & \Phi_{31,0} = 0, \\ \Phi_{12,0} = 0 & \Phi_{22,0} = 1 & \Phi_{32,0} = 0. \end{cases} \quad (4.45)$$

Moreover, the terms of $O(\varepsilon)$ give

$$O(\varepsilon) : \begin{cases} \Phi_{33,1} = 0 & \Phi_{23,1} = 0, \\ \Phi_{13,1}(t, k) = \int_0^t e^{-4ik^2(t-t')} (2kg_{02}^{(1)} + ig_{12}^{(1)})(t') dt', \\ \Phi_{11,1} = 0, \\ \Phi_{21,1} = \int_0^t \sigma e^{4ik^2(t-t')} (2k\bar{g}_{01}^{(1)} - i\bar{g}_{11}^{(1)})(t') dt', \\ \Phi_{31,1} = \int_0^t \sigma e^{4ik^2(t-t')} (2k\bar{g}_{02}^{(1)} - i\bar{g}_{12}^{(1)})(t') dt', \\ \Phi_{12,1} = \int_0^t e^{-4ik^2(t-t')} (2kg_{01}^{(1)} + ig_{11}^{(1)})(t') dt', \\ \Phi_{22,1} = 0, \quad \Phi_{32,1} = 0. \end{cases} \quad (4.46)$$

and the terms of $O(\varepsilon^2)$ give

$$O(\varepsilon^2) : \begin{cases} \Phi_{13,2} = \int_0^t e^{-4ik^2(t-t')} (2kg_{02}^{(2)} + ig_{12}^{(2)})(t') dt', \\ \Phi_{23,2} = \int_0^t \sigma [(2k\bar{g}_{01}^{(1)} - i\bar{g}_{11}^{(1)})(t')\Phi_{13,1}(t', k) + i\bar{g}_{01}^{(1)}(t')g_{02}^{(1)}(t')] dt', \\ \Phi_{33,2} = \int_0^t \sigma [(2k\bar{g}_{02}^{(1)} - i\bar{g}_{12}^{(1)})(t')\Phi_{13,1}(t', k) + i|g_{02}^{(1)}(t')|^2] dt', \\ \Phi_{11,2} = \int_0^t \left[-i\sigma(|g_{01}^{(2)}|^2 + |g_{02}^{(2)}|^2)(t') + (2kg_{01}^{(1)} + ig_{11}^{(1)})(t')\Phi_{21,1}(t', k) \right. \\ \left. + (2kg_{02}^{(1)} + ig_{12}^{(1)})(t')\Phi_{31,1}(t', k) \right] dt', \\ \Phi_{21,2} = \int_0^t \sigma e^{4ik^2(t-t')} (2k\bar{g}_{01}^{(2)} - i\bar{g}_{11}^{(2)})(t') dt', \\ \Phi_{31,2} = \int_0^t \sigma e^{4ik^2(t-t')} (2k\bar{g}_{02}^{(2)} - i\bar{g}_{12}^{(2)})(t') dt', \\ \Phi_{12,2} = \int_0^t e^{4ik^2(t-t')} (2kg_{01}^{(2)} + ig_{11}^{(2)})(t') dt', \\ \Phi_{22,2} = \int_0^t \sigma \left[(2k\bar{g}_{01}^{(1)} - i\bar{g}_{11}^{(1)})(t')\Phi_{12,1}(t', k) + i|g_{01}^{(1)}(t')|^2 \right] dt', \\ \Phi_{32,2} = \int_0^t \sigma \left[(2k\bar{g}_{02}^{(1)} - i\bar{g}_{12}^{(1)})(t')\Phi_{12,1}(t', k) + i\bar{g}_{01}^{(1)}(t')\bar{g}_{02}^{(1)}(t') \right] dt'. \end{cases} \quad (4.47)$$

Similarly, we will have the analogue formulas for $\{\phi_{ij,l}\}_{i,j=1}^3, l = 0, 1, 2$ expressed in terms of the boundary data at $x = L$, that is $\{f_{ij}^{(l)}\}_{i=0,1}^{j=1,2}, l = 1, 2$.

On the other hand, expanding (4.29) and assuming for simplicity that $m_{11}(\mathcal{A})(k)$ has no zeros, we find

$$g_{11}^{(1)}(t) = \frac{2}{\pi i} \int_{\partial D_3^0} (k\Phi_{12,1-}(t, k) + ig_{01}^{(1)}) dk - \frac{4\sigma}{\pi i} \int_{\partial D_3^0} \frac{1}{\Delta} (k\bar{\phi}_{21,1-} + i\sigma \bar{f}_{01}^{(1)}) dk, \quad (4.48a)$$

$$g_{12}^{(1)}(t) = \frac{2}{\pi i} \int_{\partial D_3^0} (k\Phi_{13,1-}(t, k) + ig_{02}^{(1)}) dk - \frac{4\sigma}{\pi i} \int_{\partial D_3^0} \frac{1}{\Delta} (k\bar{\phi}_{31,1-} + i\sigma \bar{f}_{02}^{(1)}) dk, \quad (4.48b)$$

$$f_{11}^{(1)}(t) = -\frac{2}{\pi i} \int_{\partial D_3^0} (k\phi_{12,1-}(t, k) + if_{01}^{(1)}) dk + \frac{4\sigma}{\pi i} \int_{\partial D_3^0} \frac{1}{\Delta} (k\bar{\Phi}_{21,1-} + i\sigma \bar{g}_{01}^{(1)}) dk, \quad (4.48c)$$

$$f_{12}^{(1)}(t) = -\frac{2}{\pi i} \int_{\partial D_3^0} (k\phi_{13,1-}(t, k) + if_{02}^{(1)}) dk + \frac{4\sigma}{\pi i} \int_{\partial D_3^0} \frac{1}{\Delta} (k\bar{\Phi}_{31,1-} + i\sigma \bar{g}_{02}^{(1)}) dk, \quad (4.48d)$$

We also find that

$$\begin{aligned} \Phi_{12,1-} &= 4k \int_0^t e^{-4ik^2(t-t')} g_{01}^{(1)}(t') dt', \\ \Phi_{13,1-} &= 4k \int_0^t e^{-4ik^2(t-t')} g_{02}^{(1)}(t') dt', \\ \phi_{21,1-} &= 4\sigma k \int_0^t e^{4ik^2(t-t')} \bar{f}_{01}^{(1)}(t') dt', \\ \phi_{31,1-} &= 4\sigma k \int_0^t e^{4ik^2(t-t')} \bar{f}_{02}^{(1)}(t') dt'. \end{aligned} \quad (4.49)$$

The Dirichlet problem can now be solved perturbatively as follows: assuming for simplicity that $m_{11}(\mathcal{A})(k)$ has no zeros and given $g_{01}^{(1)}, g_{02}^{(1)}$ and $f_{01}^{(1)}, f_{02}^{(1)}$, we can use equation (4.49) to determine $\Omega^{(1)}$. We can then compute $g_{11}^{(1)}, g_{12}^{(1)}$ and $f_{11}^{(1)}, f_{12}^{(1)}$ from (??) and then $\Phi_{1j,1}, j = 2, 3$ from (4.46) and the analogue results for $\phi_{j1,1}, j = 2, 3$. In the same way we can determine $\Phi_{1j,2}, j = 2, 3$ from (4.47) and the analogue results for $\phi_{j1,2}, j = 2, 3$, then compute $g_{11}^{(2)}, g_{12}^{(2)}$ and $f_{11}^{(2)}, f_{12}^{(2)}$. And these arguments can be extended to the higher order and also can be extended to the systems (4.27), (4.28) and (4.26), thus yields a constructive scheme for computing $S(k)$ to all orders. The construction of $S_L(k)$ is similar.

Similarly, these arguments also can be used to the Neumann problem. That is to say, in all cases, the system can be solved perturbatively to all orders.

4.4. The large L limit. In the limit $L \rightarrow \infty$, the representations for $g_{11}(t), g_{12}(t)$ and $g_{01}(t), g_{02}(t)$ of theorem reduce to the corresponding representations on the half-line. Indeed, as $L \rightarrow \infty$,

$$\begin{aligned} f_{01} &\rightarrow 0, & f_{02} &\rightarrow 0, & f_{11} &\rightarrow 0, & f_{12} &\rightarrow 0, \\ \phi_{ij} &\rightarrow \delta_{ij}, & \frac{\Sigma}{\Delta} &\rightarrow 1 \text{ as } k \rightarrow \infty \text{ in } D_3 \end{aligned}$$

Thus, the $L \rightarrow \infty$ limits of the representations (4.29a), (4.29b) and (4.31a), (4.31b) are

$$\begin{aligned} g_{11}(t) &= \frac{2}{i\pi} \int_{\partial D_3^0} (k\Phi_{12-} + ig_{01})dk + \frac{2}{\pi} \int_{\partial D_3^0} (g_{01}\Phi_{22-} + g_{02}\Phi_{32-})dk - \frac{1}{\pi} \int_{\partial D_3^0} g_{01}\bar{\phi}_{22-}dk \\ g_{12}(t) &= \frac{2}{i\pi} \int_{\partial D_3^0} (k\Phi_{13-} + ig_{02})dk + \frac{2}{\pi} \int_{\partial D_3^0} (g_{01}\Phi_{23-} + g_{02}\Phi_{33-})dk - \frac{1}{\pi} \int_{\partial D_3^0} g_{02}\bar{\phi}_{33-}dk. \end{aligned} \quad (4.50)$$

and

$$g_{01}(t) = \frac{1}{\pi} \int_{\partial D_3^0} \Phi_{12+} dk, \quad g_{02}(t) = \frac{1}{\pi} \int_{\partial D_3^0} \Phi_{13+} dk, \quad (4.51)$$

respectively, and these formulas coincide with the corresponding half-line formulas, see [22].

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REFERENCES

- [1] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Commun. Pure Appl. Math. 21(1968) 467-490.
- [2] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Methods for solving the Korteweg-de Vries equation, Phys. Rev. Lett. 19(1967) 1095-1097.
- [3] A. S. Fokas, A unified transform method for solving linear and certain nonlinear PDEs, Proc. R. Soc. Lond. A 453(1997), 1411-1443.

- [4] A. S. Fokas, On the integrability of linear and nonlinear partial differential equations, *J. Math. Phys.* 41(2000) 4188-4237.
- [5] A. S. Fokas, Integrable nonlinear evolution equations on the half-line, *Commun. Math. Phys.* 230(2002), 1-39.
- [6] A.S. Fokas, A Unified Approach to Boundary Value Problems, in: CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, 2008.
- [7] A.S. Fokas, A.R. Its, An initial-boundary value problem for the Korteweg-de Vries equation. *Math. Comput. Simul.* 37(1994) 293-321.
- [8] A.S. Fokas, A.R. Its, L.Y. Sung, The nonlinear Schrödinger equation on the half-line. *Nonlinearity* 18(2005) 1771-1822.
- [9] A.S. Fokas, A.R. Its, An initial-boundary value problem for the sine-Gordon equation. *Theor.Math. Physics* 92(1992) 388-403.
- [10] A.S. Fokas, A.R. Its, The linearization of the initial-boundary value problem of the nonlinear Schrodinger equation. *SIAM J. Math. Anal.* 27(1996) 738-764.
- [11] A.S. Fokas, A.R. Its, The nonlinear Schrodinger equation on the interval. *J. Phys.A* 37(2004) 6091-6114.
- [12] A. Boutet De Monvel, A.S. Fokas, D. Shepelsky, Integrable nonlinear evolution equations on a finite interval, *Comm. Math. Phys.* 263(2006) 133-172.
- [13] A. Boutet de Monvel, A.S. Fokas, D. Shepelsky, The mKDV equation on the half-line, *J. Inst. Math. Jussieu.* 3(2004), 139-164.
- [14] A. S. Forkas, J. Lenells, The unified method: I. nonlinearizable problem on the half-line, *J. Phys. A: Math. Theor.* 45(2012) 195201;
- [15] J. Lenells, A. S. Forkas, The unified method: II. NLS on the half-line t-periodic boundary conditions, *J. Phys. A: Math. Theor.* 45(2012) 195202;
- [16] J. Lenells, A. S. Forkas, The unified method: III. Nonlinearizable problem on the interval, *J. Phys. A: Math. Theor.* 45(2012) 195203;
- [17] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems: Asymptotics for the MKdV equation, *Ann. of Math.* 137(1993), 295-368.
- [18] J. Lenells, Initial-boundary value problems for integrable evolution equations with 3×3 Lax pairs, *Physica D* 241(2012) 857-875.
- [19] J. Lenells, The Degasperis-Procesi equation on the half-line, *Nonlinear Analysis* 76(2013) 122-139.
- [20] J. Xu, E. Fan, The unified transform method for the Sasa-Satsuma equation on the half-line, *Proc. R. Soc. A.* 469(2013) 20130068.

- [21] J. Xu, E. Fan, The three wave equation on the half-line, Physics Letter A, 378(2014) 26-33.
- [22] J. Xu, E. Fan, The Fokas method for the two-component nonlinear Schrödinger equation on the half-line, submitted.
- [23] J. Xu, E. Fan, The initial-boundary value problem for the Ostrovsky-Vakhnenko equation on the half-line, submitted.
- [24] S.V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, Sov. Phys. JETP, 38(1974) 248-253.
- [25] T.H. Busch, J.R. Anglin, Dark-bright solitons in inhomogeneous Bose-Einstein condensates, Phys. Rev. Lett. 87(2001) 010401.

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